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SINGULAR PERTURBATIONS AND OPTIMAL CONTROL.(U)

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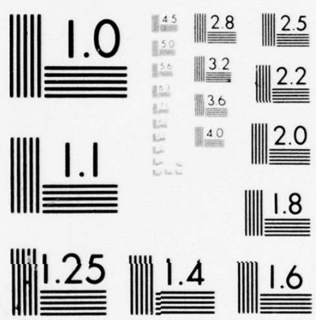

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SINGULAR PERTURBATIONS AND OPTIMAL CONTROL*

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Abstract

These lecture notes are intended to provide an elementary account of some of the recent mathematical effort in applying singular perturbations theory to optimal control problems, to demonstrate the practical importance of this asymptotic technique to current engineering studies, and to suggest several open problems needing further research. Readers are referred to the survey article by Kokotovic, O'Malley, and Sannuti for a discussion of related topics and for additional references.

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1. A Simple Singular Perturbation Problem.

Following Cole (1968), we consider the motion of a linear oscillator initially at rest, subject to an impulse of strength I_0. To find the displacement y, we need to solve the initial value problem

$$(1)\qquad m\frac{d^2y}{dt^2} + \beta\frac{dy}{dt} + ky = I_0\delta(t), \quad y(0^-) = \frac{dy}{dt}(0^-) = 0$$

where m, β, and k are the usual mass, damping, and spring constants and $\delta(t)$ is a delta function peaked at $t = 0$. For $t > 0^+$, then, we'll have

$$(2)\qquad m\frac{d^2y}{dt^2} + \beta\frac{dy}{dt} + ky = 0, \quad y(0^+) = 0, \quad \frac{dy}{dt}(0^+) = I_0/m.$$

(The last condition follows by integrating (1) from 0^- to 0^+ where the displacement remains zero.)

A regular perturbation problem would result if we sought an approximate solution on any finite t interval for (relatively) small values of the damping constant β (i.e., $\beta^2 << mk$). The weakly damped oscillator problem has a well-known solution (cf. Cole) which depends analytically on β and tends to the solution of the undamped oscillator problem as $\beta \to 0$. There is necessarily nonuniform convergence at $t = \infty$, however, since the undamped oscillator continues its motion with undiminished maximum amplitude, while the slightly damped solution ultimately decays to zero. A two-variable expansion procedure (cf. Cole or, e.g., Greenlee and Snow (1975)) is most appropriate to describe the asymptotic behavior near $t = \infty$, though we shall not pursue the matter.

A singular perturbation problem occurs when the mass m is relatively small. (If we ignored the mass, there would be a non-negligible effect on the solution.) Nondimensionalizing the initial value problem (2), let us set

$$\bar{t} = kt/\beta \quad \text{and} \quad \bar{y} = \beta y/I_0$$

to obtain the dimensionless problem

$$(3)\qquad \varepsilon\frac{d^2\bar{y}}{d\bar{t}^2} + \frac{d\bar{y}}{d\bar{t}} + \bar{y} = 0, \quad \bar{y}(0) = 0, \quad \frac{d\bar{y}}{d\bar{t}} = \frac{1}{\varepsilon}$$

with the small, positive parameter $\varepsilon = mk/\beta^2$ (i.e., we shall suppose mk is small compared to β^2, so we simultaneously examine the problem with a relatively small spring constant k or large damping constant β. We note that mk/β^2 had an infinite limit (instead of zero) in the small damping problem).

Omitting the bars, the exact solution of the initial value problem (3) is given by

$$(4)\qquad y(t,\varepsilon) = \frac{1}{\varepsilon(\rho_1 - \rho_2)}\left(e^{\rho_1 t} - e^{\rho_2 t}\right)$$

where

$$\rho_1(\varepsilon) = [-1 + \sqrt{1 - 4\varepsilon}]/2\varepsilon = -1 + O(\varepsilon)$$

and

$$\rho_2(\varepsilon) = [-1 - \sqrt{1 - 4\varepsilon}]/2\varepsilon = -\frac{1}{\varepsilon} + 1 + O(\varepsilon)$$

are the roots of the characteristic polynomial $\varepsilon\rho^2 + \rho + 1 = 0$. (Here, the Landau symbol $O(\varepsilon)$ represents an error e such that $|e| \le k\varepsilon$ for some $k > 0$ and all sufficiently small $\varepsilon > 0$.) Since $\rho_1 \to -1$ and $\rho_2 \to -\infty$ as $\varepsilon \to 0$, we have

$$(5)\qquad y(t,\varepsilon) = e^{-t} - e^{-t/\varepsilon} + O(\varepsilon)$$

since, e.g., $e^{-(\frac{1}{\varepsilon} - 1)t} = e^{-t/\varepsilon} + O(\varepsilon)$ for $t \ge 0$. [Note that the solution to the physically meaningless problem with $\varepsilon < 0$ would blow up as $\varepsilon \to 0$.] It is essential to note the nonuniform convergence of the solution which occurs at $t = 0$ as $\varepsilon \to 0^+$, i.e., we have

$$\lim_{\varepsilon \to 0} y(t,\varepsilon) = \begin{cases} 0, & t = 0 \\ e^{-t}, & t > 0 \end{cases}$$

since $e^{-t/\varepsilon}$ is asymptotically negligible for $t > 0$. Such nonuniform convergence is the hallmark of singular perturbation problems. It is also important to realize that the limiting solution

$$Y_0(t) = e^{-t} \tag{6}$$

for $t > 0$ satisfies the "reduced" equation $Y_0' + Y_0 = 0$, but neither of the initial conditions prescribed for y. Why Y_0 selects the initial value 1 is a mystery still to be explained.

Using the convergent expansions for the $\rho_i(\varepsilon)$'s as $\varepsilon \to 0$, the solution (4) of (3) is seen to be of the form

$$y(t,\varepsilon) = Y(t,\varepsilon) + \Pi(\tau,\varepsilon) \tag{7}$$

where the "outer" solution

$$Y(t,\varepsilon) = e^{\rho_1 t}/\varepsilon(\rho_1 - \rho_2)$$

has an asymptotic expansion

$$Y(t,\varepsilon) \sim \sum_{j=0}^{\infty} Y_j(t)\varepsilon^j \tag{8}$$

as $\varepsilon \to 0$, for all $t \geq 0$, usually called the outer expansion, and the "boundary layer" correction

$$\Pi(\tau,\varepsilon) = e^{\varepsilon\rho_2\tau}/\varepsilon(\rho_1 - \rho_2)$$

in the "stretched" (or boundary layer) variable

$$\tau = t/\varepsilon \tag{9}$$

has an asymptotic expansion

$$\Pi(\tau,\varepsilon) \sim \sum_{j=0}^{\infty} \Pi_j(\tau)\varepsilon^j \tag{10}$$

whose terms Π_j all tend to zero as τ tends to infinity. (Olver (1974) is an excellent reference concerning asymptotic expansions. We note that such expansions are usually divergent, rather than convergent, and that (8) should be interpreted as meaning that for any integer $N \geq 0$,

$$Y(t,\varepsilon) - \sum_{j=0}^{N} Y_j(t)\varepsilon^j = o(\varepsilon^N)$$

(i.e., the right hand side tends to zero as $\varepsilon \to 0$ at a rate faster than ε^N).) The terms of the expansions (8) and (10) can be uniquely obtained by a Taylor series expansion of $Y(t,\varepsilon)$ and $\Pi(\tau,\varepsilon)$ about $\varepsilon = 0$ or, more directly, via the straightforward procedure given below.

Since the solution (7) is asymptotically equal to the outer solution (8) for $t > 0$ (i.e., $\Pi \sim 0$), the asymptotic series for Y must satisfy

$$\varepsilon Y'' + Y' + Y = 0$$

as a power series in ε. Equating coefficients of ε^j for each $j \geq 0$ requires that $Y_j' + Y_j = -Y_{j-1}''$, so

$$Y_j(t) = e^{-t}Y_j(0) - \int_0^t e^{-(t-s)}Y_{j-1}''(s)ds, \quad j \geq 0 \tag{11}$$

with $Y_{-1}(t) \equiv 0$. Thus, the outer expansion (8) will be completely and uniquely obtained termwise up to specification of the initial value

$$Y(0,\varepsilon) \sim \sum_{j=0}^{\infty} Y_j(0)\varepsilon^j.$$

Since y and Y both satisfy the differential equation of (3) and $\frac{d^k}{dt^k} = \frac{1}{\varepsilon^k}\frac{d^k}{d\tau^k}$, the boundary layer correction Π must satisfy

$$\frac{d^2\Pi}{d\tau^2} + \frac{d\Pi}{d\tau} + \varepsilon\Pi = 0, \quad \tau \geq 0$$

as a power series in ε. Thus, $d^2\Pi_j/d\tau^2 + d\Pi_j/d\tau = -\Pi_{j-1}$ for each $j \geq 0$. Asking that $\Pi_j \to 0$ as $\tau \to \infty$, we then have

$$\Pi_j(\tau) = -e^{-\tau} \frac{d\Pi_j(0)}{d\tau} - \int_0^\tau \int_0^r e^{-(\tau-r)} \Pi_{j-1}(r)dr \tag{12}$$

so the boundary layer (correction's) expansion (10) is determined termwise up to selection of its initial derivative

$$\frac{d\Pi}{d\tau}(0,\varepsilon) \sim \sum_{j=0}^{\infty} \frac{d\Pi_j(0)}{d\tau} \varepsilon^j = - \sum_{j=0}^{\infty} \Pi_j(0)\varepsilon^j.$$

The representation (7) implies the "matching" condition

$$y'(0,\varepsilon) = \frac{1}{\varepsilon} = Y'(0,\varepsilon) + \frac{1}{\varepsilon}\frac{d\Pi}{d\tau}(0,\varepsilon).$$

Hence, we must have

$$\left\{ \begin{array}{l} \dfrac{d\Pi_0(0)}{d\tau} = 1 \\ \text{and} \\ \dfrac{d\Pi_j(0)}{d\tau} = -Y'_{j-1}(0) \quad \text{for each} \quad j \geq 1. \end{array} \right. \tag{13}$$

Thus, the initial values needed for the boundary layer correction terms are determined from earlier terms of the outer solution. In particular, $\Pi_0(\tau)$ is now completely known. Likewise, the remaining initial condition

$$y(0,\varepsilon) = 0 = Y(0,\varepsilon) + \Pi(0,\varepsilon)$$

implies that we must have

$$Y_j(0) = -\Pi_j(0) \quad \text{for each} \quad j \geq 0. \tag{14}$$

Thus, Π_0 determines Y_0 and, more generally, the terms Y_j of the outer expansion can be determined in a termwise bootstrap fashion, since $\Pi_j(0)$

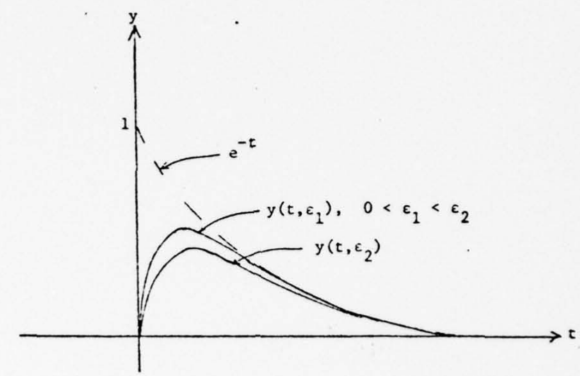
depends only on $Y'_{j-1}(0)$. Indeed, (12) and (13) imply that

$$Y_j(0) = -Y'_{j-1}(0) \quad \text{for each} \quad j \geq 1.$$

Our formal procedure, then, produces the asymptotic solution

$$y(t,\varepsilon) = (e^{-t} - e^{-\tau}) + \varepsilon\{e^{-t}(t-1) - e^{-\tau}(\tau - 1)\} + O(\varepsilon^2) \tag{15}$$

in agreement with the exact solution (4). This result for $\tau = t/\varepsilon$ clearly displays the rapid initial rise in displacement obtained for small ε, followed by an ultimate decay like a massless system. We note that the boundary layer calculation, leading to the representation of terms (12), played an essential role in obtaining the asymptotic solution $Y(t,\varepsilon)$ appropriate for $t > 0$. In particular, knowing that $\Pi_0(\tau) = -e^{-\tau}$ implied that the maximum displacement of the system tends to one as $\varepsilon \to 0$. Pictorially, we have the displacements



We note that Chapter 1 of Andronov, Vitt, and Khaikin (1966) considers the oscillator with small mass somewhat more intuitively, developing the idea of an initial jump.

For the corresponding two-point problem

$$\text{(16)} \qquad \varepsilon y'' + y' + y = 0, \quad y(0) = 0, \quad y(1) = e^{-1},$$

the unique solution is again given by

$$y(t,\varepsilon) = Y_0(t) + \Pi_0(\tau) + O(\varepsilon)$$

with Y_0 and Π_0 as before. The limiting solution $Y_0(t)$ for $t > 0$, however, now satisfies the reduced problem

$$Y_0' + Y_0 = 0, \quad Y_0(1) = e^{-1}$$

obtained by using the differential equation (16) with $\varepsilon = 0$ and canceling the initial condition. Likewise, for the initial value problem

$$\text{(17)} \qquad \varepsilon y'' + y' + y = 0, \quad y(0) = 0, \quad y'(0) = 1,$$

the limiting solution Y_0 for $t > 0$ will be the trivial solution of the reduced problem

$$Y_0' + Y_0 = 0, \quad Y_0(0) = 0.$$

Cancellation of a boundary condition to define the reduced problem is natural, since the differential equation is of first order when $\varepsilon = 0$. That no simple cancellation occurs in the solution of our oscillator problem (3) is because the boundary condition $y'(0) = 1/\varepsilon$ becomes singular as $\varepsilon \to 0^+$ (cf. O'Malley and Keller (1968), however, for the definition of an appropriate cancellation rule). Boundary value problems for scalar differential equations with small parameters multiplying the highest derivatives are one of the best studied singular perturbation problems (cf. Wasow (1965) or O'Malley (1974a)). Such problems and their generalizations do occur in control.

2. Simplified Models in Control and Systems Theory.

Consider a physical system described by the equations

$$\text{(1)} \qquad \begin{cases} \dot{x} = f(x,y,z,u,t) \\ \varepsilon\dot{y} = g(x,y,z,u,t) \\ \dot{z} = \frac{1}{\mu} h(x,y,z,u,t) \end{cases}$$

where x, y, z, and u are vectors, ε is a small positive parameter, and μ is a large positive parameter. Roughly, y corresponds to a fast-varying vector and z to a slowly-varying vector (compared to x). It would be natural to attempt to simplify the system by neglecting the small parameters ε and $1/\mu$ and solving the reduced system

$$\text{(2)} \qquad \begin{cases} \dot{X} = f(X,Y,Z,u,t) \\ 0 = g(X,Y,Z,u,t) \\ \dot{Z} = 0. \end{cases}$$

Then, we'd have

$$\text{(3)} \qquad \begin{cases} Z = \text{constant} \\ \text{and} \\ Y = \phi(X,Z,u,t) \end{cases}$$

presuming that we could find a unique root of the nonlinear equation

$$g(X,\phi,Z,u,t) = 0.$$

Thus, we'd be left with the lower-dimensional, non-"stiff" model

$$\text{(4)} \qquad \dot{X} = f(X,\phi(X,Z,u,t),Z,u,t) = F(X,Z,u,t).$$

Such approximations are common in many areas of science, e.g., an analogous procedure is known as the prompt jump approximation in nuclear reactor theory (cf. Hetrick (1971)) and as the pseudo-steady state hypothesis in enzyme kinetics (cf. Rubinow (1975)) and is basic to the development of numerical methods for integrating stiff differential equations (cf. Willoughby (1974)). We need to determine to what extent such simplifications are valid.

Desoer (1970, 1977) uses ε to indicate the degree of smallness of certain "stray" elements (e.g., stray capacitances and lead inductances) in circuits, and he uses μ to represent "sluggish" elements (like chokes and coupling capacitors). The stray elements will affect the high-frequency behavior, while the sluggish elements affect the low-frequency behavior. On finite t-intervals, we have a singular perturbation of (2) by including the ε terms of (1) and a regular perturbation by including the $1/\mu$ terms. Stability considerations for appropriate high-frequency ("boundary layer") models will be needed to justify the mid-frequency (or reduced) model (4). We shall not discuss the appropriate low-frequency approximations, noting only that they must deal with nonuniform convergence at $t = \infty$ and a regular perturbation analysis for finite t. Many other circuit theory examples are given in Andronov, Vitt, and Khaiken (1966). Sophisticated discussions of regular perturbation theory include Rellich (1969) and Kato (1966).

Since, in practice, one always neglects some small parasitics, Kokotovic has claimed that all control problems are singularly perturbed. Successful control engineers must, then, naturally use their intuition to check the hypotheses of the theorems which guarantee legitimacy to the reduced order models which they use.

Sannuti and Kokotovic (1969) gave an example of a voltage regulator described by a linear system

$$\begin{cases} \dfrac{dx}{dt} = a_1 x + a_2 z \\ \varepsilon \dfrac{dz}{dt} = a_3 z + bu \end{cases}$$

for $t \geq 0$ with

$$a_1 = 0.1 \begin{pmatrix} -2 & 5 \\ 0 & -5 \end{pmatrix}, \quad a_2 = \begin{pmatrix} 0 & 0 & 0 \\ 1.6 & 0 & 0 \end{pmatrix},$$

$$a_3 = \begin{pmatrix} -10/7 & 60/7 & 0 \\ 0 & -2.5 & -7.5 \\ 0 & 0 & -1 \end{pmatrix}, \quad \text{and} \quad b = \begin{pmatrix} 0 \\ 0 \\ 3 \end{pmatrix} .$$

for the not-so-small parameter $\varepsilon = 0.1$. Here, the object is to minimize the cost functional

$$J = \frac{1}{2} \int_0^\infty \left[x' \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} x + u^2 \right] dt.$$

Setting $\varepsilon = 0$ corresponds to neglecting the small time constants. Even though $\varepsilon = 0.1$ isn't very small, the second-order reduced problem provides an acceptable solution, which is much easier to compute than the exact solution of the full fifth-order model. (Since a_1 is also small, one might simultaneously also neglect it.) Sannuti and Kokotovic observe that it doesn't work to integrate the full system for $\varepsilon = 0.1$ with the feedback solution appropriate for $\varepsilon = 0$, but one shouldn't expect it to.

Likewise, Kokotovic and Yackel (1972) discuss a model for speed control of a small dc motor described by the state equations

$$\begin{cases} \dfrac{d\omega}{dt} = Di/G \\ \varepsilon L \dfrac{di}{dt} = -C\omega - R_a i + v. \end{cases}$$

Since the armature inductance εL is typically small, it is common to use the simplified model

$$\frac{d\Omega}{dt} = \frac{D}{GR_a} (-C\Omega + v)$$

in designing servosystems. This acknowledges the fact that mechanical time constants are large compared to electrical ones, but the model will not be appropriate for a fast initial transient. Finally, more examples are found in Chow and Kokotovic (1977) and elsewhere throughout the literature.

3. The Singularly Perturbed Linear State Regulator Problem.

a. Problem Formulation.

Consider the problem of minimizing the scalar cost functional

(1) $$J = \frac{1}{2} y'(1,\varepsilon)\Pi(\varepsilon)y(1,\varepsilon) + \frac{1}{2}\int_0^1 [y'(t,\varepsilon)Q(t,\varepsilon)y(t,\varepsilon) + u'(t,\varepsilon)u(t,\varepsilon)]dt$$

where the vector $y = \begin{pmatrix} x \\ z \end{pmatrix}$ satisfies the singularly perturbed system of state equations

(2) $$\begin{cases} \dfrac{dx}{dt} = A_1(t,\varepsilon)x + A_2(t,\varepsilon)z + B_1(t,\varepsilon)u \\ \varepsilon\dfrac{dz}{dt} = A_3(t,\varepsilon)x + A_4(t,\varepsilon)z + B_2(t,\varepsilon)u \end{cases}$$

on $0 \le t \le 1$ with prescribed initial vectors

(3) $$x(0,\varepsilon) \quad \text{and} \quad z(0,\varepsilon).$$

For symmetric, positive semi-definite matrices Π and Q and a fixed $\varepsilon > 0$, such problems have a well-known unique solution (cf. Kalman (1960)). We'd like to determine the limiting behavior of this solution as the small positive parameter ε tends towards zero.

We shall not discuss several important linear generalizations of this problem which have already been dealt with in the control literature. They include the fixed endpoint problem (cf. Wilde and Kokotovic (1973)), several parameter problems (cf. Asatani (1974)), distributed parameter systems (cf. Lions (1973)), stochastic problems (cf., e.g., Blankenship and Sachs (1977) and Haddad (1976)), and systems with small time-delays (cf. Sannuti and Reddy (1973)). We will, however, consider a nonlinear generalization.

Let us take states x, z, and the control u to be vectors of dimensions m, n, and r, respectively; assume that the matrices A_i, B_i, Q, and Π and the vectors $x(0,\varepsilon)$ and $z(0,\varepsilon)$ all have asymptotic power series expansions as $\varepsilon \to 0$; that the A_i, B_i, and Q are infinitely differentiable functions of t; and that Π has the partitioned form

(4) $$\Pi(\varepsilon) = \begin{pmatrix} \Pi_1(\varepsilon) & \varepsilon\Pi_2(\varepsilon) \\ \varepsilon\Pi_2'(\varepsilon) & \varepsilon\Pi_3(\varepsilon) \end{pmatrix}$$

with blocks having sizes compatible with the dimensions of x and z. We note, in particular, that (4) implies that the terminal cost term

(5) $$\lambda(\varepsilon) \equiv \frac{1}{2} y'(1,\varepsilon)\Pi(\varepsilon)y(1,\varepsilon)$$

of (1) depends only on the "slow" state x when $\varepsilon = 0$. We could considerably curtail the hypotheses specifying smoothness with respect to t and ε with little loss, but the nature of the asymptotic solution would be different if the restriction (4) were not assumed (cf. Glizer and Dmitriev (1975)).

Following Kalman (1960) (and, e.g., Anderson and Moore (1971) or Coppel (1975)), we introduce the Hamiltonian

(6) $$h(x,z,p,q,u,t,\varepsilon) = (x'z')Q(t,\varepsilon)\binom{x}{z} + u'u + p'(A_1x + A_2z + B_1u) + q'(A_3x + A_4z + B_2u)$$

for m and n dimensional costate vectors p and εq which satisfy the linear adjoint (or costate) equations

(7) $$\frac{dp}{dt} = -\frac{\partial h}{\partial x} \quad \text{and} \quad \varepsilon\frac{dq}{dt} = -\frac{\partial h}{\partial z}$$

and the terminal conditions

(8) $$p(1,\varepsilon) = \frac{\partial\lambda(\varepsilon)}{\partial x(1,\varepsilon)} \quad \text{and} \quad \varepsilon q(1,\varepsilon) = \frac{\partial\lambda(\varepsilon)}{\partial z(1,\varepsilon)}$$

for λ defined by (5). (Here, we've used the costate εq to compensate for the fact that dz/dt is formally of order $O(1/\varepsilon)$. We also note that the state equations (2) take the canonical form $\frac{dx}{dt} = \frac{\partial h}{\partial p}$ and $\varepsilon\frac{dz}{dt} = \frac{\partial h}{\partial q}$.) As Kalman showed, minimization of the cost functional is equivalent to minimizing the Hamiltonian. Thus, setting

$$\frac{\partial h}{\partial u} = u + B_1'p + B_2'q = 0$$

provides the unique minimum since $\partial^2 h/\partial u^2 = I_r$ is positive definite. The optimal control is therefore given by

$$u = -B_1'p - B_2'q, \tag{9}$$

so eliminating u in (2) leaves us with a linear singularly perturbed two-point boundary value problem for the states x and z and the (scaled) costates p and q.

Our linear-quadratic regulator problem has thereby been reduced to analyzing the asymptotic behavior of the $2m + 2n$ dimensional linear system

$$(10)\quad \begin{cases} \dfrac{dx}{dt} = A_1x - B_1B_1'p - A_2z - B_1B_2'q, & x(0,\varepsilon) \text{ prescribed} \\ \dfrac{dp}{dt} = -Q_1x - A_1'p - Q_2z - A_3'q, & p(1,\varepsilon) = \Pi_1(\varepsilon)x(1,\varepsilon) + \varepsilon\Pi_2(\varepsilon)z(1,\varepsilon) \\ \varepsilon\dfrac{dz}{dt} = A_3x - B_2B_1'p + A_4z - B_2B_2'q, & z(0,\varepsilon) \text{ prescribed} \\ \varepsilon\dfrac{dq}{dt} = -Q_2'x - A_2'p - Q_3z - A_4'q, & q(1,\varepsilon) = \Pi_2'(\varepsilon)x(1,\varepsilon) + \Pi_3(\varepsilon)z(1,\varepsilon) \end{cases}$$

where $Q = \begin{pmatrix} Q_1 & Q_2 \\ Q_2' & Q_3 \end{pmatrix}$.

Linear singular perturbation problems such as (10) have been well studied. We note, for example, that Harris (1973) considers linear boundary value problems for the $p + q$ dimensional system

$$(11)\quad \begin{cases} u' = A(t,\varepsilon)u + B(t,\varepsilon)v \\ \varepsilon v' = C(t,\varepsilon)u + D(t,\varepsilon)v \end{cases}$$

on $0 \le t \le 1$ under the principal assumption that the eigenvalues of the $q \times q$ matrix $D(t,0)$ have nonzero real parts throughout $[0,1]$. He shows

that such systems have a p dimensional manifold of solutions which tend to solutions of the reduced system

$$\begin{cases} U_0' = A(t,0)U_0 + B(t,0)V_0 \\ 0 = C(t,0)U_0 + D(t,0)V_0 \end{cases}$$

as $\varepsilon \to 0$. That system has p linearly independent solutions determined by

$$U_0' = (A(t,0) - B(t,0)D^{-1}(t,0)C(t,0))U_0$$

since $V_0 = -D^{-1}(t,0)C(t,0)U_0$. Moreover, if $D(t,0)$ has k, $0 \le k \le q$, stable eigenvalues, there are k linearly independent solutions of (11) which decay to zero (like $e^{-C_1t/\varepsilon}$ for some positive definite matrix C_1) as the stretched variable

$$\tau = t/\varepsilon \tag{12}$$

tends to infinity, and there will be $q - k$ linearly independent solutions which decay to zero (like $e^{-C_2(1-t)/\varepsilon}$ for some $C_2 > 0$) as

$$\sigma = (1 - t)/\varepsilon \tag{13}$$

tends to infinity. (In a sense, this theory produces a fundamental matrix for (11) which is asymptotically valid as $\varepsilon \to 0$ in $0 \le t \le 1$ (cf. Turrittin (1952)). The results are proved by integral equation techniques.) Since the general solution of (11) is a linear combination of any $p + q$ linearly independent asymptotic solutions, the behavior of any solution to (11) satisfies a three time-scale property, i.e., such an asymptotic solution must be an additive sum of functions depending on t, τ, and σ, respectively, with the function of τ (or σ) providing boundary layer behavior (i.e., nonuniform convergence as $\varepsilon \to 0$) at $t = 0$ (or $t = 1$) and the limiting solution within $(0,1)$ being a function of t which satisfies the

reduced system there. Much more complicated behavior would result if we allowed the eigenvalues of D to cross the imaginary axis or to remain on it (an exchange of stability or neutral stability).

In the problem (10), the role of $D(t,0)$ is played by the $2n \times 2n$ Hamiltonian matrix

$$G(t) = \begin{pmatrix} A_4(t,0) & -B_2(t,0)B_2'(t,0) \\ -Q_3(t,0) & -A_4'(t,0) \end{pmatrix}. \tag{14}$$

Because $J_nG = -G'J_n$ is symmetric for the symplectic matrix

$$J_n = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}, \tag{15}$$

corresponding to any eigenvalue λ of G is another eigenvalue $-\lambda$. Thus, G will be invertible and the results corresponding to those for (11) will hold provided

(H1) <u>All eigenvalues of the matrix</u> $G(t)$ <u>have nonzero real parts throughout</u> $0 \le t \le 1$.

Indeed, G will then have n stable eigenvalues and n unstable ones, so the singularly perturbed system (10) will have n linearly independent solutions which decay to zero away from $t = 0$, n others which decay to zero for $t < 1$, and $2m$ others which satisfy the reduced system corresponding to (10) in the limit $\varepsilon \to 0$. We note that (H1) relates to the factorizability of a related characteristic polynomial and to stabilizability of associated control problems (cf. Coppel (1974)).

b. <u>The Reduced Problem</u>.

The reduced system for (10) has the form

$$\begin{cases} \dfrac{d}{dt}\begin{pmatrix} X_0 \\ P_{10} \end{pmatrix} = M\begin{pmatrix} X_0 \\ P_{10} \end{pmatrix} + J_nL'J_n\begin{pmatrix} Z_0 \\ P_{20} \end{pmatrix} \\ 0 = L\begin{pmatrix} X_0 \\ P_{10} \end{pmatrix} + G\begin{pmatrix} Z_0 \\ P_{20} \end{pmatrix} \end{cases} \tag{16}$$

for

$$M = \begin{pmatrix} A_{10} & -B_{10}B_{10}' \\ -Q_{10} & -A_{10}' \end{pmatrix} \quad \text{and} \quad L = \begin{pmatrix} A_{30} & -B_{20}B_{10}' \\ -Q_{20}' & -A_{20}' \end{pmatrix},$$

where, e.g., $A_{10} = A_1(t,0)$. [Here, we've used P_{10} and P_{20}, instead of P_0 and Q_0, to represent costates, to avoid confusion between Q_0 and submatrices or expansion coefficients of Q.] It is natural to retain the limiting boundary conditions

$$X_0(0) = x(0,0) \quad \text{and} \quad P_{10}(1) = \Pi_{10}(0)X_{10}(1) \tag{17}$$

of (10) for (16), thereby defining a reduced boundary value problem. Then, however, the reduced problem (16)-(17) cannot be expected to provide the limiting solution to (10) near $t = 0$ or $t = 1$ since it fails to account for the initial condition for z or the terminal condition for q in terms of x and z. Its tremendous advantage over (10), however, is having differential order $2m$ instead of $2m + 2n$.

Since (H1) allows us to obtain $\begin{pmatrix} Z_0 \\ P_{20} \end{pmatrix}$ as a linear function of $\begin{pmatrix} X_0 \\ P_{10} \end{pmatrix}$, (16)-(17) is equivalent to the boundary value problem

$$\frac{d}{dt}\begin{pmatrix} X_0 \\ P_{10} \end{pmatrix} = V(t)\begin{pmatrix} X_0 \\ P_{10} \end{pmatrix}, \quad X_0(0) = x(0,0), \quad P_{10}(1) = \Pi_1(0)X_0(1) \tag{18}$$

for $V = M - J_mL'J_nG^{-1}L$. Further, the Hamiltonian structure of M and G^{-1} implies that of V, i.e., $J_mV = -V'J_m = J_mM + L'J_nG^{-1}L$ is symmetric, so that (18) becomes

$$\frac{d}{dt}\begin{pmatrix} X_0 \\ P_{10} \end{pmatrix} = \begin{pmatrix} V_1 & V_2 \\ V_3 & -V_1' \end{pmatrix}\begin{pmatrix} X_0 \\ P_{10} \end{pmatrix}, \quad X_0(0) = x(0,0), \quad P_{10}(1) = \Pi_1(0)X_0(1) \tag{19}$$

with

$$V_1 = A_{10} + (B_{10}B_{20}' \quad A_{20})J_nG^{-1}\begin{pmatrix} A_{30} \\ -Q_{20}' \end{pmatrix},$$

$$V_2 = -B_{10}B_{10}' - (B_{10}B_{20}' \quad A_{20})J_nG^{-1}\begin{pmatrix} B_{20}B_{10}' \\ A_{20}' \end{pmatrix} = V_2',$$

and

$$V_3 = -Q_{10} + (A_{30}' \quad -Q_{20})J_n G^{-1}\begin{pmatrix} A_{30} \\ -Q_{20}' \end{pmatrix} = V_3'.$$

Thus, the reduced problem (19) is an m^{th} order regulator problem, which we shall call the reduced regulator problem, and it is natural to seek a solution to it in the feedback form

$$P_{10} = K_{10}X_0 \tag{20}$$

where the symmetric $m \times m$ matrix K_{10} satisfies the matrix Riccati differential equation

$$\dot{K}_{10} = -K_{10}V_1 - V_1'K_{10} - K_{10}V_2K_{10} + V_3, \quad K_{10}(1) = \Pi_1(0) \tag{21}$$

(cf., again Kalman (1960)). If K_{10} exists back to $t = 0$, we only need to integrate the initial value problem

$$\dot{X}_0 = (V_1 + V_2K_{10})X_0, \quad X_0(0) = x(0,0) \tag{22}$$

to completely solve the reduced problem (16)-(17).

According to Bucy (1967), necessary and sufficient conditions to solve the linear Hamiltonian system (19) are

(H2) <u>The</u> $m \times m$ <u>matrices</u> $V_2(t)$ <u>and</u> $V_3(t)$ <u>are both negative semi-definite throughout</u> $0 \le t \le 1$.

We conjecture that (H2) is redundant. To actually calculate V_2 and V_3, we'd have to obtain the blocks of G^{-1} (for a method to do so, cf. Theorem 5 of Coppel (1974) and the calculations of O'Malley and Kung (1975)). For A_{40} invertible, O'Malley and Kung showed that $V_2 \le 0$ while L. Anderson (personal communication) has since shown that $V_3 \le 0$ then holds. Likewise, O'Malley (1975) found that (H2) held when x, z, and u are scalars.

Our Riccati solution of the reduced regulator problem suggests that the original problem could also be solved through a Riccati feedback approach and that's true (cf. Yackel and Kokotovic (1973) and O'Malley and Kung (1974)). One would set $\begin{pmatrix} p \\ q \end{pmatrix} = k(t,\varepsilon)\begin{pmatrix} x \\ z \end{pmatrix}$. That approach is nontrivial (but also important in other contexts) because the Riccati equation for k is singularly perturbed, i.e., it has a small parameter multiplying its derivative term.

Note that another (perhaps more natural) reduced problem is obtained by setting $\varepsilon = 0$ in the original optimal control problem (1)-(4). Thus, suppose we

$$(23)\quad \begin{cases} \text{minimize} \\ J = \frac{1}{2}X_0'(1)\Pi_1(0)X_0(1) + \frac{1}{2}\int_0^1 \{(X_0' \quad Z_0')Q_0(t)\begin{pmatrix} X_0 \\ Z_0 \end{pmatrix} + U_0'U_0\}dt \\ \text{with} \\ \dot{X}_0 = A_{10}X_0 + A_{20}Z_0 + B_{10}U_0, \quad X_0(0) = x(0,0) \\ 0 = A_{30}X_0 + A_{40}Z_0 + B_{20}U_0. \end{cases}$$

Here, both Z_0 and U_0 play the role of control variables, while X_0 remains a state vector. If A_{40}^{-1} exists, we can find Z_0 as a linear function of X_0 and U_0 and (23) reduces to a standard linear regulator problem in the form

$$(24)\quad \begin{cases} \text{minimize} \\ J = \frac{1}{2}X_0'(1)\Pi_1(0)X_0(1) + \frac{1}{2}\int_0^1 \{-X_0'(t)V_3(t)X_0(t) + W_0'(t)R(t)W_0(t)\}dt \\ \text{with} \\ \dot{X}_0 = V_1(t)X_0 + \mathcal{B}(t)W_0, \quad X_0(0) = x(0,0). \end{cases}$$

Here W_0 is a linear combination of U_0 and X_0; V_1, V_2, and V_3 are the submatrices of (19); and R is a positive-definite matrix such that $V_2 = -\mathcal{B}R^{-1}\mathcal{B}'$. The equivalence of the reduced regulator problems (19) and

(24) follows under the hypotheses of O'Malley and Kung (1975), but we expect it to be generally true under hypothesis (H1).

c. Boundary Layers.

Since the matrix $G(t)$ has n stable eigenvalues and n unstable ones, the general theory for linear singularly perturbed boundary value problems and some experience suggest that we seek an asymptotic solution to our two-point problem (10) in the form

$$(25)\quad \begin{cases} x(t,\varepsilon) = X(t,\varepsilon) + \varepsilon m_1(\tau,\varepsilon) + \varepsilon n_1(\sigma,\varepsilon) \\ z(t,\varepsilon) = Z(t,\varepsilon) + m_2(\tau,\varepsilon) + n_2(\sigma,\varepsilon) \\ p(t,\varepsilon) = P_1(t,\varepsilon) + \varepsilon\rho_1(\tau,\varepsilon) + \varepsilon\gamma_1(\sigma,\varepsilon) \\ q(t,\varepsilon) = P_2(t,\varepsilon) + \rho_2(\tau,\varepsilon) + \gamma_2(\sigma,\varepsilon) \end{cases}$$

where the outer expansion

$$(26)\quad (X,Z,P_1,P_2) \sim \sum_{j=0}^{\infty} (X_j,Z_j,P_{1j},P_{2j})\varepsilon^j$$

provides the asymptotic solution to (10) within $(0,1)$; the initial boundary layer correction

$$(27)\quad (\varepsilon m_1,m_2,\varepsilon\rho_1,\rho_2) \sim \sum_{j=0}^{\infty} (\varepsilon m_{1j},m_{2j},\varepsilon\rho_{1j},\rho_{2j})\varepsilon^j$$

satisfies (10) and its terms tend to zero as the stretched variable

$$\tau = t/\varepsilon$$

tends to infinity; and the terms of the terminal boundary layer correction

$$(28)\quad (\varepsilon n_1,n_2,\varepsilon\gamma_1,\gamma_2) \sim \sum_{j=0}^{\infty} (\varepsilon n_{1j},n_{2j},\varepsilon\gamma_{1j},\gamma_{2j})\varepsilon^j$$

tend to zero as

$$\sigma = (1 - t)/\varepsilon$$

tends to infinity. In part, we write these forms of the asymptotic solution to display its three time scale structure and the relative importance of the different scales to the different components of the asymptotic solution. In practice, one would typically compute only the ε^0 and ε^1 coefficients. We further note that the control relation (9) and the representation (25) imply that the optimal control will have a *corresponding* asymptotic representation

$$(29)\quad u(t,\varepsilon) = U(t,\varepsilon) + v(\tau,\varepsilon) + w(\sigma,\varepsilon)$$

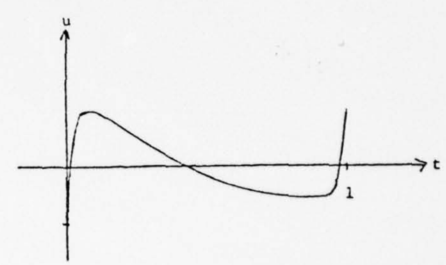
where, e.g.,

$$v(\tau,\varepsilon) = -\varepsilon B_1'(\varepsilon\tau,\varepsilon)\rho_1(\tau,\varepsilon) - B_2'(\varepsilon\tau,\varepsilon)\rho_2(\tau,\varepsilon)$$

and the boundary layer corrections v and w are asymptotically significant only near the endpoints $t = 0$ and $t = 1$, respectively. Since u has the form

$$u(t,\varepsilon) = U_0(t) + v_0(\tau) + w_0(\sigma) + O(\varepsilon)$$

the optimal control will generally converge nonuniformly near each endpoint and a boundary layer analysis is necessary to determine the endpoint control. A typical plot of optimal control is pictured in the figure.



Finally, the expansions (25) and (29) imply that optimal cost will have the

form

$$J^*(\varepsilon) \sim \sum_{k=0}^{\infty} J_k \varepsilon^k \tag{30}$$

where the leading term J_0 is the optimal cost for the reduced regulator problem (19), i.e.,

$$J_0 = \frac{1}{2} x'(0,0)K_{10}(0)x(0,0) \tag{31}$$

(cf. (20) and Kalman (1960)). The boundary layer contributions to the cost, like the integral $\int_0^1 e^{-t/\varepsilon}dt$, are $O(\varepsilon)$.

We must now learn how to calculate the asymptotic solution (25). Since the boundary layer correction terms become negligible within $(0,1)$, the outer expansion (26) must satisfy the differential system of (10) as a power series in ε. The leading terms $(X_0, Z_0, P_{10}, P_{20})$ will necessarily satisfy the limiting system (16) and, by the form of (25), the boundary conditions (17). [Unlike the spring-mass system, then, the boundary conditions appropriate for the limiting solution here are obtained without first calculating a boundary layer term.] Under hypotheses (H1) and (H2), the resulting reduced problem (16)-(17) has a unique solution. Higher order terms in (26) will then satisfy nonhomogeneous forms of (16)-(17) with successively known forcing terms. The Fredholm alternative, then, guarantees that they, too, will have unique solutions.

Since the outer solution (26) accounts termwise for the initial condition for x and the terminal condition for p, the initial boundary layer correction (27) must adjust for any "boundary layer jump" $z(0,\varepsilon) - Z(0,\varepsilon)$ while the terminal correction (28) must account for the terminal condition for q. (We recall that Z_0 and P_{20} were determined from algebraic equations.) Since the solution of (10) will be asymptotically the sum of the outer expansion (26) and the initial boundary layer correction (27) near $t = 0$ (σ being asymptotically infinite), while (26) satisfies (10), it follows that (27) must satisfy (10) as a function of τ. Thus, *we seek a decaying solution* of the linear system

$$\tag{32} \begin{cases} \dfrac{dm_1}{d\tau} = \varepsilon A_1(\varepsilon\tau,\varepsilon)m_1 + A_2(\varepsilon\tau,\varepsilon)m_2 - \varepsilon B_1(\varepsilon\tau,\varepsilon)B_1'(\varepsilon\tau,\varepsilon)\rho_1 \\ \qquad - B_1(\varepsilon\tau,\varepsilon)B_2'(\varepsilon\tau,\varepsilon)\rho_2 \\ \dfrac{d\rho_1}{d\tau} = \varepsilon Q_1(\varepsilon\tau,\varepsilon)m_1 - Q_2(\varepsilon\tau,\varepsilon)m_2 - \varepsilon A_1'(\varepsilon\tau,\varepsilon)\rho_1 - A_3'(\varepsilon\tau,\varepsilon)\rho_2 \\ \dfrac{dm_2}{d\tau} = \varepsilon A_3(\varepsilon\tau,\varepsilon)m_1 + A_4(\varepsilon\tau,\varepsilon)m_2 - \varepsilon B_2(\varepsilon\tau,\varepsilon)B_1'(\varepsilon\tau,\varepsilon)\rho_1 \\ \qquad - B_2(\varepsilon\tau,\varepsilon)B_2'(\varepsilon\tau,\varepsilon)\rho_2 \\ \dfrac{d\rho_2}{d\tau} = -\varepsilon Q_2(\varepsilon\tau,\varepsilon)m_1 - Q_3(\varepsilon\tau,\varepsilon)m_2 - \varepsilon A_2'(\varepsilon\tau,\varepsilon)\rho_1 - A_4'(\varepsilon\tau,\varepsilon)\rho_2 \end{cases}$$

for $\tau \geq 0$ satisfying the initial condition

$$m_2(0,\varepsilon) \sim z(0,\varepsilon) - Z(0,\varepsilon). \tag{33}$$

In particular, then, for $\varepsilon = 0$ we have the limiting constant coefficient *boundary layer problem*

$$\tag{34} \begin{cases} \dfrac{dm_{10}}{d\tau} = A_{20}(0)m_{20} - B_{10}(0)B_{20}'(0)\rho_{20} \\ \dfrac{d\rho_{10}}{d\tau} = -Q_{20}(0)m_{20} - A_{30}'(0)\rho_{20} \\ \dfrac{dm_{20}}{d\tau} = A_{40}(0)m_{20} - B_{20}(0)B_{20}'(0)\rho_{20}, \quad m_{20}(0) = z(0,0) - Z_0(0) \\ \dfrac{d\rho_{20}}{d\tau} = -Q_{30}(0)m_{20} - A_{40}'(0)\rho_{20}. \end{cases}$$

Presuming, then, that we can find an exponentially decaying solution to the initial value problem

$$\frac{d}{d\tau}\begin{pmatrix} m_{20} \\ \rho_{20} \end{pmatrix} = G(0)\begin{pmatrix} m_{20} \\ \rho_{20} \end{pmatrix}, \quad m_{20}(0) = z(0,0) - Z_0(0) \tag{35}$$

we'll determine the remaining decaying terms as

(36) $$(m_{10}(\tau),\ \rho_{10}(\tau)) = -\int_{\tau}^{\infty}\left(\frac{dm_{10}(s)}{d\tau},\ \frac{d\rho_{10}(s)}{d\tau}\right) ds$$

since $dm_{10}/d\tau$ and $d\rho_{10}/d\tau$ are linear combinations of m_{20} and ρ_{20}. We recall that $G(0)$ has half its eigenvalues stable and half unstable. Thus, the decaying solutions of (35) are spanned by n linearly independent quasipolynomial solutions of the form

(37) $$s_i(\tau)e^{\lambda_{i0}\tau}, \quad i = 1, 2, \ldots, n$$

where the s_i's are polynomials in τ and the λ_{i0}'s are stable eigenvalues of $G(0)$ (cf., e.g., Coddington and Levinson (1955)), i.e., we must have

$$\begin{pmatrix} m_{20} \\ \rho_{20} \end{pmatrix} = \sum_{i=1}^{n} s_i(\tau)e^{\lambda_{i0}\tau}k_i$$

for n appropriate vectors k_i. Let us assume

(H3a) <u>The</u> $n \times n$ <u>matrix</u> $T_{10} \equiv (s_{11}(0) \quad s_{21}(0) \quad \cdots \quad s_{n1}(0))$ <u>is nonsingular where the</u> n-<u>vectors</u> $s_{j1}(0)$ <u>are such that</u>

$$s_j(0) = \begin{pmatrix} s_{j1}(0) \\ s_{j2}(0) \end{pmatrix}, \quad j = 1, \ldots, n.$$

Then, the solution of (35) is uniquely given by

(38) $$\begin{pmatrix} m_{20} \\ \rho_{20} \end{pmatrix} = \left(s_1(\tau)e^{\lambda_{10}\tau} \quad s_2(\tau)e^{\lambda_{20}\tau} \quad \cdots \quad s_n(\tau)e^{\lambda_{n0}\tau}\right) \cdot T_{10}^{-1}(z(0,0) - Z_0(0)).$$

We note that (H3a) is independent of the basis (37) chosen for the space of decaying solutions. Higher order terms in the boundary layer expansion (27) will also follow uniquely in turn since they will satisfy a nonhomogeneous version of the problem (34) with successively known, exponentially decaying forcing terms.

An alternative reformulation of the initial value problem (35) could be

obtained by setting

(39) $$\rho_{20} = Km_{20}$$

where K is a constant symmetric solution of the algebraic Riccati problem

(40) $$KA_{40}(0) + A_{40}'(0)K - KB_{20}(0)B_{20}'(0)K + Q_{30}(0) = 0.$$

This is, of course, natural once we recognize (35) as an infinite interval n^{th} order regulator problem which we shall call the initial boundary layer regulator. Assuming appropriate stabilizability-dectectability assumptions (cf., e.g., Kucera (1972) or (1973)) would provide a unique positive semi-definite matrix K for which the remaining initial value problem for

(41) $$\frac{dm_{20}}{d\tau} = (A_{40}(0) - B_{20}(0)B_{20}'(0)K)m_{20}, \quad \tau \geq 0$$

has a decaying solution. These hypotheses would, of course, be equivalent to (H3a) and somewhat weaker than the boundary layer controllability and observability assumptions of Wilde and Kokotovic (1973). Because the origin is a saddle point for (35), direct numerical integration of (35) should be avoided. (One couldn't help but excite exponentially growing modes.) Numerical solution of (40)-(41) would be highly preferable. An alternative would be to follow Coppel (1974)'s use of diagonalization of (35) via a nonsingular, symplectic matrix.

Proceeding analogously, we find that the terminal boundary layer correction (28) must satisfy the system (10) as a function of σ. Moreover, the boundary condition for q implies that it must also satisfy

(42) $$\begin{aligned} Y_2(0,\varepsilon) - \Pi_3(\varepsilon)\eta(0,\varepsilon) &\sim \Pi_2'(\varepsilon)(X(1,\varepsilon) + \varepsilon n_1(0,\varepsilon)) \\ &\quad + \Pi_3(\varepsilon)Z(1,\varepsilon) - P_2(1,\varepsilon). \end{aligned}$$

Continuing as before, we find that the leading terms will be a decaying solution of the system

$$\frac{d}{d\sigma}\begin{pmatrix} n_{20} \\ \gamma_{20} \end{pmatrix} = -G(1)\begin{pmatrix} n_{20} \\ \gamma_{20} \end{pmatrix} \tag{43}$$

and

$$\begin{cases} \dfrac{dn_{10}}{d\sigma} = -A_{20}(1)n_{20} + B_{10}(1)B_{20}'(1)\gamma_{20} \\[2ex] \dfrac{d\gamma_{10}}{d\sigma} = Q_{20}(1)n_{20} + A_{30}'(1)\gamma_{20} \end{cases} \tag{44}$$

while $\gamma_{20}(0) - \Pi_3(0)n_{20}(0)$ is determined by the limiting outer solution. Again, (43) can be solved as a terminal boundary layer regulator and (44) would be integrated directly. We could also relate γ_{20} and n_{20} through the solution $\bar{K}$ of an algebraic Riccati equation (cf. Sannuti (1974)). Instead, let us take

$$r_i(\sigma)e^{-\lambda_{i1}\sigma}, \quad i = 1, 2, \ldots, n \tag{45}$$

to be n linearly independent decaying solutions to (43) as $\sigma \to \infty$ corresponding to eigenvalues λ_{i1} of $G(1)$ with positive real parts. Then, we shall assume

(H3b) <u>The</u> $n \times n$ <u>matrix</u> $R_{21} - \Pi_3(0)R_{11}$ <u>is nonsingular where</u>

$$\begin{pmatrix} R_{11} \\ R_{21} \end{pmatrix} = (r_1(0) \quad r_2(0) \quad \ldots \quad r_n(0)),$$

and it follows that the decaying solution of the initial value problem for (43) is uniquely given by

$$\begin{pmatrix} n_{20}(\sigma) \\ \gamma_{20}(\sigma) \end{pmatrix} = \begin{pmatrix} r_1(\sigma)e^{-\lambda_{11}\sigma} & r_2(\sigma)e^{-\lambda_{21}\sigma} & \ldots & r_n(\sigma)e^{-\lambda_{n1}\sigma} \end{pmatrix} \cdot (R_{21} - \Pi_3(0)R_{11})^{-1}(\Pi_2'(0)X_0(1) + \Pi_3(0)Z_0(1) - P_{20}(1)). \tag{46}$$

Further terms also follow without difficulty.

d. <u>Further Observations</u>.

Under hypotheses (H1)-(H3), we have been able to formally obtain the terms of the asymptotic expansions (25) for the solution of the two-point problem (10) for the states and costates. The procedure is completely justified by the asymptotic theory for such problems (cf., e.g., Harris (1973) or Vasil'eva and Butuzov (1973)). To provide an independent proof, one would need to show that the difference between the solution and the $N + 1$ term approximation formally generated is $o(\epsilon^N)$ uniformly throughout $0 \le t \le 1$.

If the hypotheses used are not satisfied, the asymptotic solution to (10) will generally not have the form (25) and the limiting solution within $(0,1)$ may not satisfy the reduced problem (16)-(17).

As a boundary value problem for a linear system of dimension $2m + 2n$, (10) will have a unique solution for each fixed $\epsilon > 0$ if a determinant of that size is nonzero. Under our hypotheses, we've been able to essentially construct a fundamental matrix valid as $\epsilon \to 0$ and expand that determinant as an asymptotic power series in ϵ (cf. O'Malley and Keller (1968)). The leading term of that determinant becomes the nonzero product of a $2m^{th}$ order determinant (nonzero by (H2)) and two n^{th} order determinants (nonzero by (H3)). Assumption (H1) guarantees that any limiting solution within $(0,1)$ will satisfy the reduced system (16), although without (H2) and (H3), it could blow up like, e.g., $\epsilon^{-\kappa}$, $\kappa > 0$. As long as G remained nonsingular, we could define the reduced system (16), but if $G(t)$ had purely imaginary eigenvalues, the solution might be rapidly oscillating (like $\sin t/\epsilon$) as $\epsilon \to 0$. A different type of asymptotic analysis, combined with boundary layer theory, would then be required (cf., e.g., Hoppensteadt and Miranker (1976)).

Under our hypotheses, it is relatively easy to write explicit expressions for the limiting solutions

$$\begin{cases} x(t,\epsilon) = X_0(t) + O(\epsilon) \\ z(t,\epsilon) = Z_0(t) + m_{20}(\tau) + n_{20}(\sigma) + O(\epsilon) \\ u(t,\epsilon) = U_0(t) + v_0(\tau) + w_0(\sigma) + O(\epsilon) \\ \text{and} \\ J^*(\epsilon) = J_0 + O(\epsilon). \end{cases} \tag{47}$$

Indeed, it is convenient to refer to X_0 as the "slow state" x_s, with the corresponding Riccati matrix K_{10} (cf. (20)) called the slow Riccati gain K_s. Together these determine the slow variables $z_s = Z_0$, $u_s = U_0$, and $J_s = J_0$. Likewise, we can call Ξ_{20} the fast initial state z_{f0} and K (cf. (40)) the fast initial (boundary layer) Riccati gain K_{f0}. Then, we'll have the corresponding fast initial control $u_{f0}(\tau) = v_0(\tau) = -B_2'(0)K_{f0}z_{f0}(\tau)$. Doing the same for the fast terminal transients, we'd rewrite (47) as

$$(48)\qquad \begin{cases} x(t,\varepsilon) = x_s(t) + O(\varepsilon) \\ z(t,\varepsilon) = z_s(t) + z_{f0}(\tau) + z_{f1}(\sigma) + O(\varepsilon) \\ u(t,\varepsilon) = u_s(t) + u_{f0}(\tau) + u_{f1}(\sigma) + O(\varepsilon) \\ \text{and} \\ J^*(\varepsilon) = J_s + O(\varepsilon). \end{cases}$$

Since dz/dt and du/dt are $O(1/\varepsilon)$ near the endpoints, while $dx/dt = O(1)$ everywhere, our earlier reference to z and u as fast-variables and to x as a slow-variable is justified.

This three-time scale separation (cf. Chow and Kokotovic (1976) and Chow (1977)) is valuable for design purposes. It reflects the intuitively desirable idea of a three stage design process consisting of a slow system (i.e., the m^{th} order regulator problem ((19) for $0 \le t \le 1$) improved by two separate fast systems (i.e., the n^{th} order boundary layer regulator problems (35) and (43) which are infinite interval problems in the stretched variables $\tau = t/\varepsilon$ and $\sigma = (1-t)/\varepsilon$, respectively). The fast systems correct the lower dimensional slow system at the endpoints $t = 0$ and $t = 1$. The time-scale separation becomes more apparent after a preliminary transformation of the system (10) to diagonal form (cf. Chang (1972)), but some interaction between time scales (as in our construction of the formal solution to (25)) is needed to analyze higher order approximate solutions.

We note that some care must be exercised in applying these results. Wilde and Kokotovic (1973), for example, observe that if A_{40} is not stable, difficulties could result if an asymptotic approximation to the optimal control is inserted into the state equations (2) and the result is integrated.

This relates to the usual problem of sensitivity regarding open loop control, but the difficulty can be avoided by only using the asymptotic formulas already developed or by combining open and closed loop control as Wilde and Kokotovic suggest.

Many practical problems concerning the use of singular perturbation theory for such regulator problems remain unanswered. Since our results are asymptotic, how small should ε be in order to use these results? If some time constants are much smaller than others, should we instead use a more refined model like the system

$$\begin{cases} \dfrac{dx}{dt} = A_1(t,\varepsilon,\mu)x + A_2(t,\varepsilon,\mu)z + A_3(t,\varepsilon,\mu)w + B_1(t,\varepsilon,\mu)u \\ \varepsilon\dfrac{dz}{dt} = A_4(t,\varepsilon,\mu)x + A_5(t,\varepsilon,\mu)z + A_6(t,\varepsilon,\mu)w + B_2(t,\varepsilon,\mu)u \\ \varepsilon\mu\dfrac{dw}{dt} = A_7(t,\varepsilon,\mu)x + A_8(t,\varepsilon,\mu)z + A_9(t,\varepsilon,\mu)w + B_3(t,\varepsilon,\mu)u\ ? \end{cases}$$

Neglecting to raise further important questions, we are nonetheless relatively content with our conclusions which we now summarize.

<u>Theorem</u>

<u>For the problem</u> (1)-(4), <u>suppose</u>

(i) <u>All eigenvalues of the Hamiltonian matrix</u>

$$G(t) = \begin{pmatrix} A_{40} & -B_{20}B_{20}' \\ -Q_{30} & -A_{40}' \end{pmatrix}$$

<u>have nonzero real parts throughout</u> $0 \le t \le 1$,

(ii) <u>the reduced (or outer)</u> m^{th} <u>order regulator problem</u>

$$\begin{cases} \dfrac{d}{dt}\begin{pmatrix} X_0 \\ P_{10} \end{pmatrix} = \begin{pmatrix} V_1 & V_2 \\ V_3 & -V_1' \end{pmatrix}\begin{pmatrix} X_0 \\ P_{10} \end{pmatrix} \\ X_0(0) = x(0,0), \quad P_{10}(1) = \Pi_1(0)X_0(1) \end{cases}$$

<u>has a unique solution</u>,

(iiia) <u>the initial</u> n^{th} <u>order boundary layer regulator</u>

$$\frac{d}{d\tau}\begin{pmatrix} m_{20} \\ \rho_{20} \end{pmatrix} = G(0)\begin{pmatrix} m_{20} \\ \rho_{20} \end{pmatrix}$$

has a unique decaying solution on $0 \le \tau < \infty$ for any initial value $m_{20}(0)$,

(iiib) the terminal n^{th} order boundary layer regulator

$$\frac{d}{d\sigma}\begin{pmatrix} n_{20} \\ \gamma_{20} \end{pmatrix} = -G(1)\begin{pmatrix} n_{20} \\ \gamma_{20} \end{pmatrix}$$

has a unique decaying solution on $0 \le \sigma < \infty$ for any value $\gamma_{20}(0) - \Pi_3(0)n_{20}(0)$.

Then, the problem has a unique solution with the optimal control, the corresponding trajectories, and the optimal cost being of the form

$$\begin{cases} u(t,\epsilon) = U(t,\epsilon) + v(\tau,\epsilon) + w(\sigma,\epsilon) \\ x(t,\epsilon) = X(t,\epsilon) + \epsilon m_1(\tau,\epsilon) + \epsilon n_1(\sigma,\epsilon) \\ z(t,\epsilon) = Z(t,\epsilon) + m_2(\tau,\epsilon) + n_2(\sigma,\epsilon) \\ J^*(\epsilon) \sim \sum_{k=0}^{\infty} J_k \epsilon^k \end{cases}$$

in $0 \le t \le 1$ where the terms all have asymptotic power series expansions and the functions of $\tau = t/\epsilon$ or $\sigma = (1 - t)/\epsilon$ decay to zero as that variable tends to infinity.

4. The Nonlinear Singularly Perturbed Regulator Problem.

a. Introduction.

Several investigators, including McIntyre (1977) whom we shall largely follow, have considered the nonlinear problem of minimizing the scalar cost

$$J = \lambda(x(1),z(1),\epsilon) + \int_0^1 \Lambda(x(t),z(t),u(t),t,\epsilon)dt \tag{1}$$

subject to the state constraints

$$\begin{cases} \dot{x} = f(x,z,u,t,\epsilon), & x(0) \text{ given} \\ \epsilon\dot{z} = g(x,z,u,t,\epsilon), & z(0) \text{ given} \end{cases} \tag{2}$$

where x, z, and u are vectors of dimensions m, n, and r, respectively, and ϵ is a small positive parameter.

Necessary conditions for optimality follow under mild assumptions, e.g., from the Pontryagin Maximum Principle and corresponding variational arguments. If we define the Hamiltonian

$$h = \Lambda + p'f + q'g \tag{3}$$

where the costates p and ϵq satisfy

$$\begin{cases} \dot{p} = -h_x, & p(1,\epsilon) = \lambda_x(x(1),z(1),\epsilon) \\ \epsilon\dot{q} = -h_z, & \epsilon q(1,\epsilon) = \lambda_z(x(1),z(1),\epsilon) \end{cases} \tag{4}$$

and the optimality condition becomes

$$h_u = 0 \tag{5}$$

while minimization will require the Legendre-Clebsch condition

$$h_{uu} \ge 0,$$

i.e., this hessian matrix must be positive semi-definite at least locally.

To provide a bounded terminal value for q, we'll ask that λ be a slowly-varying function of the fast state variable z, i.e.,

$$\lambda(x,z,\varepsilon) = \theta(x,\varepsilon z,\varepsilon). \tag{6}$$

Then, the terminal cost λ will depend only on the slow state x at $\varepsilon = 0$, so that both z and u will play the role of control vectors in the reduced control problem obtained by setting $\varepsilon = 0$ in (1) and (2).

Our results for the linear problem suggest that it is more convenient to introduce the Hamiltonian

$$H(\psi,\zeta,u,t,\varepsilon) = h(x,z,u,p,q,t,\varepsilon) \tag{7}$$

for the vectors

$$\psi = \begin{pmatrix} x \\ p \end{pmatrix} \quad \text{and} \quad \zeta = \begin{pmatrix} z \\ q \end{pmatrix}$$

of dimensions $2m$ and $2n$ respectively. The necessary conditions for optimality then reduce to a two-point boundary value problem for the nonlinear singularly perturbed system

$$\begin{cases} \dot{\psi} = J_m H_\psi(\psi,\zeta,u,t,\varepsilon) \\ \varepsilon\dot{\zeta} = J_n H_\zeta(\psi,\zeta,u,t,\varepsilon) \\ 0 = H_u(\psi,\zeta,u,t,\varepsilon) \end{cases} \tag{8}$$

where $J_k = \begin{pmatrix} 0 & I_k \\ -I_k & 0 \end{pmatrix}$, $H_{uu} \geq 0$, and $x(0)$ and $z(0)$ are prescribed while $p(1,\varepsilon) = \theta_x(x(1,\varepsilon),\varepsilon z(1,\varepsilon),\varepsilon)$ and $q(1,\varepsilon) = \frac{\partial\theta}{\partial(\varepsilon z)}(x(1,\varepsilon),\varepsilon z(1,\varepsilon),\varepsilon)$. Henceforth, then, we shall restrict attention to the asymptotic solution of the boundary value problem for (8).

b. The Strong Legendre-Clebsch Condition.

Let us assume the strong form of the Legendre-Clebsch condition, i.e.,

$$H_{uu} > 0. \tag{9}$$

(We shall not be cautious in defining a limit to the region where the hessian is positive-definite, nor to limiting the smoothness of functions in (8), though we realize that such restrictions would be of practical importance.) Under (9), the implicit function theorem implies that we can uniquely solve the optimality condition $H_u = 0$ for

$$u = \eta(\psi,\zeta,t,\varepsilon). \tag{10}$$

Thus (8) reduces to the singularly perturbed boundary value problem

$$\begin{cases} \dot{\psi} = F(\psi,\zeta,t,\varepsilon) \equiv J_m H_\psi(\psi,\zeta,\eta,t,\varepsilon) \\ \varepsilon\dot{\zeta} = G(\psi,\zeta,t,\varepsilon) \equiv J_n H_\zeta(\psi,\zeta,\eta,t,\varepsilon) \end{cases} \tag{11}$$

together with the $2m + 2n$ separated boundary conditions inherited from (2), (4) and (6). [If $H_{uu} > 0$ only held locally several roots $u = \eta$ might be possible, leading to different two-point problems (11).]

Regrettably, no adequate theory is available for such singularly perturbed two-point problems. For initial value problems, Tikhonov developed a theory under the assumption that all eigenvalues of G_ζ are locally stable (cf. Vasil'eva and Butuzov (1973)). One would expect to be able to solve appropriate boundary value problems when G_ζ has a fixed number of stable eigenvalues and a fixed number of unstable ones. We find, however, a need to restrict nonlinearities. One might, for example, expect the solution of the two-point problem

$$\begin{cases} \dot{u} = v, \quad u(0) = 0, \quad u(1) = 1 \\ \varepsilon\dot{v} = -v - v^3 \end{cases}$$

to converge to the limit $(U_0, V_0) = (1,0)$ for $t > 0$, just as the limiting solution of the initial value problem

$$\begin{cases} \dot{u} = v, & u(0) = 0 \\ \varepsilon\dot{v} = -v - v^3, & v(0) = 1 \end{cases}$$

is $(U_0, V_0) = (0,0)$. One can easily show, however, (cf. Coddington and Levinson (1952)) that the first problem has no solution for ε small. Such examples have effectively limited most singular perturbation analysis to problems like (11) with G linear or quadratic in the fast-variable ζ.

If a limiting solution to our control problem exists, we can nonetheless expect it to satisfy the reduced problem

$$(12)\qquad \begin{cases} \dot{\Psi}_0 = F(\Psi_0, Z_0, t, 0) \\ 0 = G(\Psi_0, Z_0, t, 0) \\ X_0(0) = x(0,0), \quad P_0(1) = \theta_x(X_0(1), 0, 0) \end{cases}$$

where $\Psi_0 = \begin{pmatrix} X_0 \\ P_0 \end{pmatrix}$ and $Z_0 = \begin{pmatrix} \bar{Z}_0 \\ Q_0 \end{pmatrix}$. Since the initial condition for z and the terminal condition for q have been neglected, we will generally need boundary layers (i.e., nonuniform convergence of the solution) near both the endpoints $t = 0$ and $t = 1$.

Corresponding to any root

$$(13)\qquad Z_0 = \xi(\Psi_0, t)$$

of $G(\Psi_0, \xi, t, 0) = 0$, we obtain a reduced regulator problem

$$(14)\qquad \begin{cases} \dot{\Psi}_0 = F(\Psi_0, t) \equiv F(\Psi_0, \xi(\Psi_0, t), t, 0) \\ X_0(0) = x(0,0), \quad P_0(1) = \theta_x(X_0(1), 0, 0). \end{cases}$$

It would be natural to assume

(H-a) <u>the reduced regulator problem</u> (14) <u>has a unique solution</u> <u>for</u> $0 \le t \le 1$.

(We recall that the corresponding hypotheses (H2) for the linear problem was far more concrete and easily verified.)

We note that the important matrix G_ζ is given by $G_\zeta = J_n(H_{\zeta\zeta} + H_{\zeta u}\frac{d\eta}{d\zeta})$. Moreover, differentiation of $H_u \equiv 0$ implies that $H_{uu}\frac{\partial\eta}{\partial\zeta} + H_{u\zeta} = 0$, so $H_{\zeta u} = H'_{u\zeta}$ implies that

$$(15)\qquad G_\zeta = J_n(H_{\zeta\zeta} - H'_{u\zeta}H_{uu}^{-1}H_{u\zeta})$$

and the symmetry of $J_n G_\zeta$ implies that the eigenvalues of G_ζ occur in pairs $\pm\lambda$. We can therefore guarantee that the $2n \times 2n$ matrix G has n stable eigenvalues and n unstable ones if we assume that for all arguments ψ, ζ

(H-b) G_ζ <u>has no purely imaginary eigenvalues on</u> $0 \le t \le 1$ <u>for</u> $\varepsilon = 0$.

This strong assumption implies that G_ζ is nonsingular, so there would be only one root Z_0 of $G = 0$. If it held only locally, more roots would be possible.

We might now proceed as for the linear problem and seek an asymptotic solution to our two-point boundary value problem (11) in the form

$$(16)\qquad \begin{cases} \psi(t,\varepsilon) = \Psi(t,\varepsilon) + \varepsilon\psi_L(\tau,\varepsilon) + \varepsilon\psi_R(\sigma,\varepsilon) \\ \zeta(t,\varepsilon) = Z(t,\varepsilon) + \zeta_L(\tau,\varepsilon) + \zeta_R(\sigma,\varepsilon) \end{cases}$$

where all terms have power series expansions in ε and the functions of the stretched variables

$$\tau = t/\varepsilon \quad \text{or} \quad \sigma = (1 - t)/\varepsilon$$

decay to zero as that stretched variable tends to infinity. Corresponding expansions for the optimal control and optimum cost will follow through (10) and (1), respectively. Such expansions can be shown (under hypotheses H-a, b, and c(below)) to provide the unique asymptotic solution when f and g are linear in z and u and Λ is quadratic in these variables (cf., e.g., O'Malley (1974b)). They may be valid more generally, though the appropriate stretched variables may be

$$\tau' = t/e^{-1/\varepsilon} \quad \text{and} \quad \sigma' = (1-t)/e^{-1/\varepsilon}$$

when the two-point problem (8) is quadratic in ζ (cf. Visik and Lyusternik (1960)).

When (16) holds, the solution will be asymptotically provided by the outer expansion

$$(\Psi(t,\varepsilon), Z(t,\varepsilon)) \sim \sum_{j=0}^{\infty} (\Psi_j, Z_j)\varepsilon^j \tag{17}$$

within $(0,1)$. It will therefore satisfy the full system (11) as a power series in ε. The leading term (Ψ_0, Z_0) will satisfy the reduced problem (12) and, under hypotheses (H-a,b) be the unique solution of (13)-(14). The next term will satisfy the linear problem obtained by linearization of (14) about $(\Psi_0, Z_0, t, 0)$, viz.

$$\left\{\begin{array}{l} \dot{\Psi}_1 = F_{0\psi}\Psi_1 + F_{0\zeta}Z_1 + F_{0\varepsilon} \\ \dot{Z}_0 = G_{0\psi}\Psi_1 + G_{0\zeta}Z_1 + G_{0\varepsilon} \\ \text{with} \\ X_1(0) \quad \text{and} \quad P_1(1) - \theta_{xx}(X_0(1),0,0)X_1(1) \quad \text{known successively} \end{array}\right.$$

where, e.g., $F_{0\psi} = \frac{\partial F}{\partial \psi}(\Psi_0, Z_0, t, 0)$. Since $G_{0\zeta}$ is invertible, we can solve the second equation for Z_1 leaving a linear regulator problem

$$\dot{\Psi}_1 = W_0\Psi_1 + w_0 \tag{18}$$

for Ψ_1 with

$$W_0 \equiv F_{0\psi} - F_{0\zeta}G_{0\zeta}^{-1}G_{0\psi} = \frac{\partial F}{\partial \psi}(\Psi_0, t).$$

Using the definitions of F and G plus (15), further manipulation implies that

$$W_0 = J_m\{(H_{\psi\psi} - H'_{u\psi}H_{uu}^{-1}H_{u\psi})$$
$$+ (H_{\psi\zeta} - H'_{u\psi}H_{uu}^{-1}H_{u\zeta})'(H_{\zeta\zeta} - H'_{u\zeta}H_{uu}^{-1}H_{u\zeta})^{-1}(H_{\psi\zeta} - H'_{u\psi}H_{uu}^{-1}H_{u\zeta})\}$$

and the symmetry of $J_m W_0$ implies that W_0 has the form

$$W_0 = \begin{pmatrix} W_{11} & W_{12} \\ W_{21} & -W'_{11} \end{pmatrix} \tag{19}$$

for symmetric matrices W_{12} and W_{21}. By hypothesis (H-a), the linearized homogeneous problem corresponding to (14) has a unique solution. Thus, the Fredholm alternative implies that the same is true for (18). On the other hand, at least for $\theta_{xx}(X_0(1),0,0)$ positive semidefinite, Bucy (1967) suggests that uniqueness of (18) implies that W_{12} and W_{21} are negative semidefinite and (18) could be solved through a matrix Riccati feedback.

Near $t = 0$, σ is asymptotically infinite, so we have $\psi \sim \Psi + \varepsilon\psi_L$, $\frac{d\psi}{dt} \sim \frac{d\Psi}{dt} + \frac{d\psi_L}{d\tau}$, etc. The nonlinear system (11) then implies that the initial boundary layer correction $\begin{pmatrix} \varepsilon\psi_L \\ \zeta_L \end{pmatrix}$ must be a decaying solution of

$$\left\{\begin{array}{l} \frac{d\psi_L}{d\tau} = F(\Psi + \varepsilon\psi_L, Z + \zeta_L, \varepsilon\tau, \varepsilon) - F(\Psi, Z, \varepsilon\tau, \varepsilon) \\ \text{and} \\ \frac{d\zeta_L}{d\tau} = G(\Psi + \varepsilon\psi_L, Z + \zeta_L, \varepsilon\tau, \varepsilon) - G(\Psi, Z, \varepsilon\tau, \varepsilon) \end{array}\right. \tag{20}$$

with

$$z_L(0,\varepsilon) \sim z(0,\varepsilon) - \bar{Z}(0,\varepsilon) \tag{21}$$

for $\zeta_L = \begin{pmatrix} z_L \\ q_L \end{pmatrix}$ split in half. Thus, this boundary layer will account for the boundary value for z. When $\epsilon = 0$, then, we seek a decaying solution of the system

$$(22)\quad \begin{cases} \dfrac{d\psi_{L0}}{d\tau} = F(\Psi_0(0),\ Z_0(0) + \zeta_{L0},\ 0,0) - F(\Psi_0(0), Z_0(0), 0,0) \\ \dfrac{d\zeta_{L0}}{d\tau} = G(\Psi_0(0),\ Z_0(0) + \zeta_{L0},\ 0,0) \\ \text{with} \\ z_{L0}(0) = z(0,0) - \tilde{Z}_0(0) \end{cases}$$

since $G(\Psi_0(0), Z_0(0), 0,0) = 0$. Presuming we can find an exponentially decaying solution ζ_{L0} of the second system, it will determine $d\psi_{L0}/d\tau$ and thereby

$$\psi_{L0}(\tau) = -\int_\tau^\infty \frac{d\psi_{L0}(s)}{d\tau}\, ds.$$

We are now left with a nonlinear infinite interval boundary layer regulator problem for ζ_{L0}, viz.

$$(23)\quad \begin{cases} \dfrac{d\zeta_{L0}}{d\tau} = G(\Psi_0(0),\ Z_0(0) + \zeta_{L0},\ 0,0) - G(\Psi_0(0), Z_0(0), 0,0), \quad \tau \geq 0 \\ z_{L0}(0) \text{ prescribed.} \end{cases}$$

Here, $\zeta_{L0} = 0$ is a rest point of the system and G_ζ has (under hypothesis (H-b)) n stable eigenvalues and n unstable ones. The origin is a saddle point for the system and we face a classical problem of conditional stability. Standard theory (cf., e.g., Coddington and Levinson (1955)) implies that there is an n dimensional manifold of initial values $\zeta_{L0}(0)$ (near the origin) resulting in a decaying solution to (23). We need hypotheses which guarantee that the prescribed n vector $z_{L0}(0)$ lies on this manifold for some choice of the last components $q_{L0}(0)$ of $\zeta_{L0}(0)$ (cf. Levin (1957) who first encountered such problems in singular perturbations). This is guaranteed by

our hypothesis (H3-a) for the linear problem and such an assumption may also be reasonable for nonlinear problems which have (very) small "boundary layer jumps" $|z(0,0) - \tilde{Z}_0(0)|$ (cf. Hadlock (1973)). To avoid this restrictive assumption for corresponding boundary value problems, Vasil'eva and Butuzov (1973) require the stable manifold be describable in the form

$$q_{L0} = \phi^L(z_{L0})$$

for some function ϕ^L. This reduces the problem (23) to an n^{th} order initial value problem for z_{L0} for which the origin is asymptotically stable. Other possible approaches include Liapunov functions (cf. Habets (1974)). Analogous, but linear, problems occur in obtaining higher order terms $\begin{pmatrix} \psi_{Lj} \\ \zeta_{Lj} \end{pmatrix}$, $j > 0$, and the problems recur in obtaining the terminal boundary layer correction $\begin{pmatrix} \epsilon\psi_R(\sigma,\epsilon) \\ \zeta_R(\sigma,\epsilon) \end{pmatrix}$.

Thus, we ask

(H-c) <u>the initial boundary layer regulator</u>

$$\frac{d\zeta_{L0}}{d\tau} = G(\Psi_0(0),\ Z_0(0) + \zeta_{L0},\ 0,0), \quad z_{L0}(0) \text{ prescribed}$$

<u>has a unique decaying solution for</u> $\tau \geq 0$, <u>and the terminal boundary layer regulator</u>

$$\frac{d\zeta_{R0}}{d\sigma} = -G(\Psi_0(1),\ Z_0(1) + \zeta_{R0},\ 1,0), \quad q_{R0}(0) = \frac{\partial\theta}{\partial(\epsilon z)}(X_0(1),0,0) - Q_0(1)$$

<u>has a unique decaying solution for</u> $\sigma \geq 0$.

As mentioned earlier, these are the correct assumptions when specialized to the quasilinear problem. Much further work needs to be done, even for the restricted problem when the strong Legendre-Clebsch condition holds.

c. The Weak Legendre-Clebsch Condition.

A more difficult, but more interesting, problem occurs if we let H_{uu} become singular. Then, we might have multiple solutions $u = \eta_i(\psi,\xi,t,\varepsilon)$ of the optimality condition $H_u = 0$ and we can anticipate the possibility of switching back and forth between them. (This would seem especially likely to occur if $\det(H_{uu}) = 0$ at isolated points where one root loses stability to another.) One would expect the states and costates to have corners at these interior transition points of the control, similar to behavior with bang-bang control (cf. Freedman and Kaplan (1977) and Kokotovic and Haddad (1975)). More impulsive control would lead to jumps in the states and costates. If $H_{uu} \equiv 0$ along a trajectory, a singular arc occurs and further necessary conditions for optimality can be obtained by differentiation (cf. Bell and Jacobson (1975)).

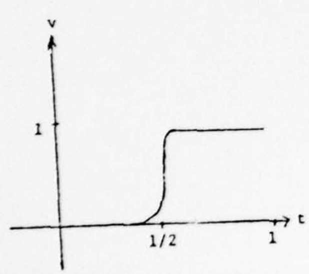
Simple singular perturbation problems featuring such discontinuous solutions include

$$\begin{cases} \dot{u} = v, \quad u(0) = 0, \quad u(1) = 1/2 \\ \varepsilon\dot{v} = v - v^3. \end{cases}$$

This problem (almost like the Coddington and Levinson nonexistence example cited above) has the limiting solution

$$u(t) = \begin{cases} 0, & 0 \le t \le 1/2 \\ t - 1/2, & 1/2 \le t \le 1 \end{cases}$$

so that v jumps discontinuously at $t = 1/2$ as $\varepsilon \to 0$. We note that the reduced problem is satisfied everywhere, but the root changes at $t = 1/2$. Pictorially, we have



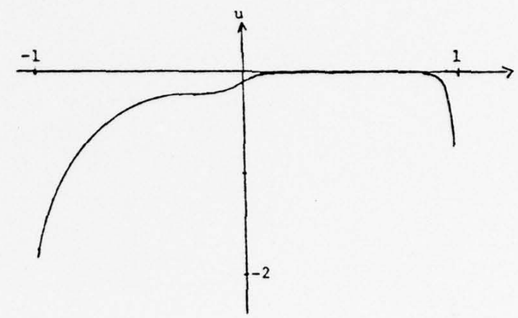
The example

$$\begin{cases} \dot{u} = v, \quad -1 \le t \le 1, \quad u(-1) = -2, \quad u(1) = -1 \\ \varepsilon\dot{v} = -tv^2 + u \end{cases},$$

can be shown (cf. Howes (1977)) to have a unique limiting solution

$$u(t) = \begin{cases} -(\sqrt{2} - 1 + \sqrt{-t})^2, & -1 \le t < 0 \\ 0 \quad , & 0 < t < 1 \end{cases}$$

with $v = \dot{u}$. We note that the reduced system $tv^2 = u$ is satisfied for $t \ne 0, 1$ and that the behavior at $t = 0$ is reminiscent of jump phenomena which occurs at gas dynamical shocks and for bang-bang control. There is also an ordinary boundary layer at $t = 1$. For small ε, the solution is as in the figure.



Studying several roots of $H_u = 0$ simultaneously recalls the geometric theories of Levinson (1951) and Andronov, Vitt, and Khaiken (1966) for higher order generalizations of the van der Pol oscillator, Fife's recent work on "transition layers" occurring as stationary patterns for reaction-diffusion systems (cf. Fife (1976)), and Carpenter's work using isolating blocks for autonomous systems to provide existence theorems and asymptotic limits to the Fitzhugh-Nagumo and other nerve impulse equations (cf. Carpenter (1977)).

The need for a global analysis of such problems is clear, both to study the stable solution manifolds involved and their regions of attraction. It would seem profitable to specialize the study to singularly perturbed Hamiltonian systems, rather than more general boundary value problems. The most complete discussion for nonlinear problems is contained in Vasil'eva and Butuzov (1973), but that is inadequate for our problem. One new approach, differential inequalities, has been successfully used by Howes (1976, 1978) to obtain existence and asymptotic behavior for scalar problems $\epsilon\ddot{y} = F(t,y,\dot{y},\epsilon)$ with F quadratic in $\dot{y}$. Likewise, Chueh, Conley, and Smoller (1977) have used invariant rectangles to obtain comparison theorems for higher order systems. It remains much easier to ask relevant questions than to answer them.

5. Cheap Control Theory and Singular Arcs.

a. The General Problem.

Now consider a linear regulator problem with cheap control, i.e., let's seek to minimize

$$(1) \qquad J(\epsilon) = \frac{1}{2}\int_0^1 [x'Q(t)x + \epsilon^2 u'R(t)u]dt$$

where ϵ is a small positive parameter subject to the (initial value problem) constraint

$$(2) \qquad \dot{x} = A(t)x + B(t)u, \quad 0 \le t \le 1, \quad x(0) \text{ prescribed.}$$

We shall take Q and R to be symmetric positive semidefinite and positive definite matrices, respectively, and we shall assume that Q, R, A, and B are infinitely differentiable functions of t. Our presentation will rely heavily on joint work with Antony Jameson.

Such problems arise naturally in a variety of contexts. Kwakernaak and Sivan (1972), Wonham (1974), Kwakernaak (1976), and Francis and Glover (1977) were concerned with the asymptotic location of poles for closed loop systems; Friedland (1971) and Moylan (1974) examined the limiting possibilities for filters; Kalman (1964) and Moylan and Anderson (1973) utilized related problems to study inverse optimal control problems; and Lions (1973) studied analogous cheap control problems (i.e., those for which the control is cheap relative to the state in the cost functional) for distributed parameter systems.

When $\epsilon = 0$, (1)-(2) becomes a singular problem of optimal control. Our asymptotic analysis of (1)-(2) allows study of the singular problem as the (nonuniform) limit of the nearly singular problems with small $\epsilon > 0$. Indeed, it provides an important new tool for analyzing singular optimal control problems (cf. Bell and Jacobson (1975)). This idea was used earlier in the singular control literature by Jacobson and coworkers to both theoretical and numerical advantage (cf. Jacobson and Speyer (1970), Jacobson, Gershwin, and

Lele (1970), and Coppel (1975)) and is analogous to the common usage of artificial viscosity techniques in fluid dynamics (cf., e.g., Richtmyer and Morton (1967)).

For each $\varepsilon > 0$, standard theory (cf., e.g., Anderson and Moore (1971)) implies that the optimal control is given by

$$u = -\frac{1}{\varepsilon^2} R^{-1} B' p \tag{3}$$

where the costate vector p satisfies the system adjoint to that for x. [The equation (3) suggests a more general asymptotic study of high gain control systems via singular perturbation methods (cf. Young, Kokotovic, and Utkin (1977) and O'Malley (1977)).] Eliminating u from the state equation, then, results in the linear, singularly perturbed boundary value problem

$$\left\{\begin{array}{l} \varepsilon^2 \dot{x} = \varepsilon^2 A x - B R^{-1} B' p, \quad x(0) \text{ prescribed} \\ \dot{p} = -Qx - A'p, \quad p(1) = 0. \end{array}\right. \tag{4}$$

Because $\varepsilon^2 A$ is singular when $\varepsilon = 0$, the familiar singular perturbation theory cited earlier (as in Harris (1973)) does not apply. One might transform the problem to a (higher dimensional) problem where those theorems are applicable. This was done by O'Malley and Jameson (1975, 1977) using transformations like those found in the earlier control papers of Moylan and Moore (1971) or Friedland (1971). We'll present a more direct solution here.

To anticipate our general results, consider the simplest such scalar problem with

$$\left\{\begin{array}{l} J(\varepsilon) = \frac{1}{2}\int_0^1 (x^2 + \varepsilon^2 u^2)\,dt \\ \dot{x} = u, \quad x(0) = 1. \end{array}\right. \tag{5}$$

The exact solution of the resulting two-point problem provides the optimal trajectory

$$x(t,\varepsilon) = (e^{-t/\varepsilon} + e^{-1/\varepsilon} e^{-(1-t)/\varepsilon})/(1 + e^{-2/\varepsilon})$$

and the optimal control $u = \dot{x}$. Since $e^{-1/\varepsilon}$ is asymptotically negligible as $\varepsilon \to 0$, we have

$$(x(t,\varepsilon),\ u(t,\varepsilon)) \approx (e^{-t/\varepsilon},\ -\frac{1}{\varepsilon} e^{-t/\varepsilon}).$$

Both functions tend to zero as $\varepsilon \to 0$ for $t > 0$, but they converge nonuniformly at $t = 0$. Indeed, $\frac{1}{\varepsilon} e^{-t/\varepsilon}$ behaves like a delta function *peaked* at $t = 0$ in the limit $\varepsilon \to 0$, while $e^{-t/\varepsilon}$ behaves like $1 - H$ for the Heaviside function H on $t \geq 0$ as $\varepsilon \to 0$. The initial impulse drives the state to zero instantaneously, in agreement with the intuitive answer $u = -\delta$ long known (cf. Ho (1972)). We note that the optimal cost satisfies

$$J^*(\varepsilon) \approx \int_0^1 e^{-2t/\varepsilon}\,dt = O(\varepsilon),$$

i.e., it is asymptotically cost free.

More generally, we can expect the solution to our two-point problem (4) to feature endpoint regions of nonuniform convergence, with convergence elsewhere to a solution of the reduced equation

$$B R^{-1} B' P_0 = 0.$$

Indeed, it is well known that the solution to the corresponding singular control problem follows a (usually lower dimensional) singular arc (along which $B'P_0 = 0$). If B^{-1} exists, $P_0 = 0$ and, then, if Q^{-1} exists, $X_0 = 0$. Otherwise, as in bifurcation theory (cf. Cesari (1975)), an auxiliary equation is needed to determine a unique limiting solution. Here $B'P_0 = 0$ restricts P_0 to a lower dimensional space. The resulting nonuniform convergence is, of course, the hallmark of singular perturbation problems. Our singular perturbation analysis, however, provides us with the unique limiting solution along the singular arc as well as the appropriate impulse at $t = 0$ to get us onto this arc.

The results we shall obtain differ fundamentally in a hierarchy of Cases. Thus, we define *Case L*, $L \geq 1$, by requiring that

$$(6) \qquad \begin{cases} B_j'QB_j = 0, \quad j = 0, 1, \ldots, L-2 \\ \text{and} \\ B_{L-1}'QB_{L-1} > 0 \end{cases}$$

throughout $0 \le t \le 1$ where

$$(7) \qquad B_0 = B \quad \text{and} \quad B_j = AB_{j-1} - \dot{B}_{j-1}, \quad j \ge 1.$$

This corresponds to the usual definition of singular arcs of order L (cf. Gon (1966) and Robbins (1967)). There are, of course, problems between cases, ones where the case changes with t, and ones beyond all cases (such as $Q \equiv 0$). Nonetheless, in Case L, we find that the optimal control takes the form

$$(8) \qquad u(t,\varepsilon) = U(t,\mu) + \frac{1}{\mu^L} v(\tau,\mu) + w(\sigma,\mu)$$

while the corresponding trajectory is like

$$(9) \qquad x(t,\varepsilon) = X(t,\mu) + \frac{1}{\mu^{L-1}} m(\tau,\mu) + \mu n(\sigma,\mu)$$

where the series are in powers of

$$\mu = \varepsilon^{1/L}$$

and the stretched variables providing endpoint boundary layers are

$$\tau = t/\mu \quad \text{and} \quad \sigma = (1-t)/\mu.$$

The corrections v and m (and w and n) tend to zero as τ (and σ) tend to infinity. It's most important to note that the optimal control features an initial impulse

$$\frac{1}{\mu^L} \sum_{j=0}^{L-1} v_j(\tau)\mu^j$$

which we'll find behaves like a linear combination of matrix impulse functions $\delta, \delta', \ldots, \delta^{(L-1)}$ with δ behaving like the asymptotic limit of a matrix $\frac{C}{\mu} e^{-Ct/\mu}$ as $\mu \to 0$ for a positive definite matrix C. Such an impulse will allow a rapid transfer from the given initial state $x(0)$ in n-space to $X(0^+,0)$ on a singular arc lying on a manifold of dimension $n - Lr$. The limiting control $U(t,0)$ within $(0,1)$ is that corresponding to the singular arc solution. At $t = 1$, the control will converge nonuniformly, but it will not be impulse-like. We note that the trajectory will feature impulsive behavior at $t = 0$ whenever $L > 1$ and that the optimal cost $J^*(\mu)$ will have an asymptotic power series expansion in μ whose limit $J_0 = \frac{1}{2}\int_0^1 X'(t,0)Q(t)X(t,0)dt$ is the cost of following the singular arc solution. For more details, see O'Malley and Jameson (1977).

b. Case 1.

We'll now limit attention to Case 1 problems where

$$(10) \qquad B'QB > 0$$

in $[0,1]$, noting that it is the only Case with bounded state x at $t = 0$. (A different frequency domain condition for "bounded peaking" is given by Francis and Glover (1977)). Since a straightforward presentation of the state-costate solution is given by O'Malley (1976), we'll instead seek a solution with $p = kx$ so that the optimal control will be in the feedback form

$$(11) \qquad u = -\frac{1}{\varepsilon^2} R^{-1}B'kx$$

where the matrix k is the unique symmetric, positive semi-definite solution of the $n \times n$ Riccati terminal value problem

$$(12) \qquad \varepsilon^2\dot{k} + \varepsilon^2(kA + A'k + Q) = kBR^{-1}B'k, \quad k(1,\varepsilon) = 0.$$

(Note that Jameson and O'Malley (1975) discussed the corresponding algebraic Riccati problem. A somewhat different discussion of singularly perturbed Riccati equations is contained in Womble, Potter, and Speyer (1976).)

Past experience suggests that we seek a solution for the Riccati gain k in the form

$$k(t,\varepsilon) = K(t,\varepsilon) + \varepsilon\ell(\sigma,\varepsilon) \tag{13}$$

where the outer solution K has a power series solution

$$K(t,\varepsilon) \sim \sum_{j=0}^{\infty} K_j(t)\varepsilon^j$$

and the boundary layer correction $\varepsilon\ell$ satisfies

$$\ell(\sigma,\varepsilon) \sim \sum_{j=0}^{\infty} \ell_j(\sigma)\varepsilon^j$$

with the terms all tending to zero as

$$\sigma = (1 - t)/\varepsilon$$

tends to infinity. Thus, K must satisfy

$$\varepsilon^2\dot{K} + \varepsilon^2(KA + A'K + Q) = KBR^{-1}B'K \tag{14}$$

for $t < 1$ and its limit K_0 must satisfy the reduced equation $K_0BR^{-1}B'K_0 = 0$, so that

$$B'K_0 = 0. \tag{15}$$

In the unlikely event that B is square and nonsingular, we have the unique solution $K_0 \equiv 0$. Otherwise, (15) merely restricts K_0 to lie in the null space of B'. Indeed, (15) makes (14) a "singular" singular-perturbation problem, beyond the reach of standard techniques for singularly perturbed initial value problems (cf. Vasil'eva (1963) or O'Malley (1974a)). One might also proceed by making explicit use of the matrix pseudoinverse (cf., e.g., Campbell (1976) and Campbell, Meyer, and Rose (1976)).

We shall manipulate the equation (14) for K. Postmultiplying by B provides an equation for $(KB)^{\cdot}$. Premultiplying by B' and equating coefficients of ε^2 then imply that

$$(B'K_1B)R^{-1}(B'K_1B) = B'QB > 0$$

and this allows us to solve for $B'KB \sim \varepsilon B'K_1B > 0$. Thus, $(B'KB)^{-1} = O(\frac{1}{\varepsilon})$ is taken to be positive-definite and

$$B'K = \varepsilon^2R(B'KB)^{-1}[B_1'K + B'Q + (B'K)^{\cdot} + B'KA] \tag{16}$$

where $B_1 = AB - \dot{B}$. Backsubstituting into (14) finally yields the substitute equation

$$\begin{aligned}\dot{K} + KA + A'K + Q = [KB_1 + QB + (KB)^{\cdot} + A'KB][B'QB + B'(KB)^{\cdot} \\ + B'KB_1 + B'A'KB]^{-1}[B_1'K + B'Q + (B'K)^{\cdot} + B'KA].\end{aligned} \tag{17}$$

Differential equations for successive terms K_j now follow by equating coefficients successively in (17).

When $\varepsilon = 0$, we obtain the parameter-free Riccati equation

$$\dot{K}_0 + K_0A_1 + A_1'K_0 + Q_1 = K_0S_1K_0, \quad K_0(1) = 0 \tag{18}$$

with

$$A_1 = A - B_1(B'QB)^{-1}B'Q,$$

$$Q_1 = Q - QB(B'QB)^{-1}B'Q,$$

and

$$S_1 = B_1(B'QB)^{-1}B_1' \geq 0.$$

(We note that this equation is well known (cf. Moylan and Moore (1971)).)

Introducing

$$P_1 = I_n - B(B'QB)^{-1}B'Q, \tag{19}$$

we readily find that $Q_1 = P_1'QP_1 \geq 0$, so the standard linear regulator theory implies the existence of a unique positive semi-definite solution to (18).

Since $B'K_0 = 0$ must hold along solutions of (18), one might wonder whether K_0 is overdetermined. To clarify the situation, we introduce

$$P_2 = I_n - P_1 = B(B'QB)^{-1}B'Q, \tag{20}$$

noting that P_1 and P_2 are projections $(P_i^2 = P_i)$ such that

$$B'QP_1 = 0, \quad P_1B = 0, \quad \text{and} \quad P_1P_2 = 0.$$

Thus, P_1 maps into the null space $N(B'Q)$ of $B'Q$, P_2 into the range $R(B)$ of B, P_1' into $N(B')$, and P_2' into $R(QB)$. (In the special case that P_1 is symmetric, (20) implies a direct sum decomposition of n-space.) We note that (15) implies that

$$P_2'K_0 = 0 \tag{21}$$

so the symmetric matrix K_0 satisfies

$$K_0 = P_1'K_0 = K_0P_1 = P_1'K_0P_1. \tag{22}$$

Thus, (18) is actually a terminal value problem for $P_1'K_0$ and the limiting problem isn't overdetermined. Because P_1 is usually singular, (18) is essentially a lower order differential equation for $P_1'K_0P_1$ in the null space of B' (cf. the analogous discussion of an algebraic Riccati equation by Kwatny (1974)).

Higher order terms K_j satisfy linearized versions of the problem for

K_0. Thus, (16) implies a linear algebraic equation for $B'K_j$ (and thereby $P_2'K_j$), while (17) provides a nonhomogeneous linear differential equation for $P_1'K_j = P_1'K_jP_1$. All that needs to be prescribed termwise is a terminal value

$$P_1'(1)K(1,\varepsilon)P_1(1) \sim \sum_{j=0}^{\infty} P_1'(1)K_j(1)P_1(1)\varepsilon^j \tag{23}$$

(the first term necessarily being the zero matrix). Splitting the problem up termwise into an algebraic equation for $P_2'K$ and a differential equation for $P_1'K$, corresponds to the frequent use of auxiliary and bifurcation equations in a complementary fashion.

Because $B'K_1B > 0$ while $k(1,\varepsilon) = 0$, the outer solution must be corrected to order $O(\varepsilon)$ in a boundary layer near $t = 1$. This suggested the representation (13). Since the system (12) is satisfied by both k and the outer solution K, (13) implies that the boundary layer correction $\varepsilon\ell$ must be a decaying solution of the nonlinear system

$$\frac{d\ell}{d\sigma} = -\frac{1}{\varepsilon}(\ell BR^{-1}B'K + KBR^{-1}B'\ell) + \varepsilon(\ell A + A'\ell) - \ell BR^{-1}B'\ell \tag{24}$$

for $\sigma \geq 0$. In particular, ℓ_0 must satisfy

$$\begin{aligned}\frac{d\ell_0}{d\sigma} = &-\ell_0B(1)R^{-1}(1)B'(1)\ell_0 - \ell_0B(1)R^{-1}(1)B'(1)K_1(1)\\ &- K_1(1)B(1)R^{-1}(1)B'(1)\ell_0.\end{aligned} \tag{25}$$

Further, $B'(1)\ell_0(0) = -B'(1)K_1(1)$ is known in terms of

$$C_0 = R^{-1/2}(1)B'(1)K_1(1)B(1)R^{-1/2}(1) > 0$$

(cf. (16)). Indeed, it provides the unique decaying solution of (25),

$$\ell_0(\sigma) = -2R^{-1/2}(1)B'(1)\ell_0(0)(I_n + e^{2C_0\sigma})^{-1}C_0^{-1}\ell_0'(0)B(1)R^{-1/2}(1), \tag{26}$$

i.e., $P_1'(1)\ell_0(\sigma)$ is determined in terms of $P_2'(1)\ell_0(\sigma)$ and $B'(1)\ell_0(\sigma)$.

Further decaying terms ℓ_j follow successively as solutions of linear equations, with the needed initial value $B'(1)\ell_j(0) = -B'(1)K_j(1)$ known through lower order terms of the outer expansion.

The optimal trajectory must satisfy the linear initial value problem

(27) $$\varepsilon^2\dot{x} = (\varepsilon^2 A - BR^{-1}B'(K + \varepsilon\ell))x, \quad x(0) \text{ given.}$$

Thus $B'K_0 = 0$ implies that the corresponding reduced system will be the linear equation $BR^{-1}B'K_1X_0 = 0$, so we can expect the limiting trajectory to satisfy

(28) $$B'K_1X_0 = 0.$$

The corresponding singular arc trajectory must therefore lie in the null space of $B'K_1$ (a space of rank r since we're in Case 1). Because $x(0)$ won't generally lie on this lower dimensional manifold, an initial boundary layer correction of the state is required at $t = 0$. Another boundary layer is needed at $t = 1$, due to the nonuniform convergence of the coefficient ℓ there. Thus we are led to seeking a trajectory of the form

(29) $$x(t,\varepsilon) = X(t,\varepsilon) + m(\tau,\varepsilon) + n(\sigma,\varepsilon)$$

for endpoint boundary layers m and n. Details of that expansion are contained in the references.

c. Related Problems.

For the preceding cheap control problem, both the outer solution for the Riccati gain (in reverse time) and for the state were initial value problems for singular singular-perturbation problems, i.e., systems of the form

(30) $$\varepsilon\dot{y} = f(t,y,\varepsilon), \quad y(0) \text{ given}, \quad 0 \le t \le 1$$

where the jacobian

$$f_y(t,y,0)$$

is singular. We shall consider such problems for m-vectors y with infinitely differentiable coefficients under the assumption

(H) the matrix $f_y(t,y,0)$ has a constant rank k, $0 \le k < m$, for all t and y; its nonzero eigenvalues are all stable; and its null space is spanned by $m - k$ linearly independent eigenvectors.

Then, the $k = m$ problem (which can be solved using Tikhonov's results (cf. O'Malley (1974a))) suggests a solution in the form

(31) $$y(t,\varepsilon) = Y(t,\varepsilon) + \Pi(\tau,\varepsilon)$$

where the boundary layer correction Π becomes asymptotically negligible as $\tau = t/\varepsilon$ tends to infinity. Further, we can expect the limiting solution $Y_0(t)$ for $t > 0$ to be a solution of the reduced problem

(32) $$f(t,Y_0,0) = 0,$$

presuming it is consistent (otherwise, we cannot expect a bounded limiting solution).

For simplicity, consider only the nearly linear problem where

(33) $$f(t,y,\varepsilon) = F(t)y + G(t) + \varepsilon h(t,y,\varepsilon).$$

Then, hypothesis (H) guarantees the existence of a smooth orthogonal matrix E such that

(34) $$EF = \begin{bmatrix} U \\ 0 \end{bmatrix}$$

is row-reduced and of rank k for every t. (E can be readily obtained in terms of the singular value decomposition of F and numerically via Householder transformations.) Splitting E as

$$E = \begin{bmatrix} E_1 \\ E_2 \end{bmatrix} \tag{35}$$

(after the k^{th} row) $P = E_1'E$ will be a projection such that $R(P) = R(F)$ and $Q = E_2'E_2$ will be a complementary projection such that $R(Q) = N(F')$ and $S = E_1FE_1'$ is a stable $k \times k$ matrix (cf. O'Malley and Flaherty (1977) and O'Malley (1978)).

Defining

$$z = Ey = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} E_1y \\ E_2y \end{pmatrix}, \tag{36}$$

we get a new initial value problem

$$\begin{cases} \varepsilon\dot{z}_1 = Sz_1 + E_1FE_2'z_2 + E_1G + \varepsilon\tilde{h}_1(z_1,z_2,t,\varepsilon), & z_1(0) = E_1(0)y(0) \\ \dot{z}_2 = \frac{1}{\varepsilon}E_2G + \tilde{h}_2(z_1,z_2,t,\varepsilon), & z_2(0) = E_2(0)y(0) \end{cases} \tag{37}$$

to which we can apply the Tikhonov theory provided

$$E_2G = 0. \tag{38}$$

That holds if and only if the (formal) reduced equation $FY_0 + G = 0$ is consistent. [Like the usual procedure, we seek to transform the singular singular-perturbation problem to a regular one.]

A more natural approach is through power series. Since the representation (31) implies that the outer solution $Y(t,\varepsilon)$ must satisfy

$$F(t)Y = -G(t) + \varepsilon[\dot{Y} - h(Y,t,\varepsilon)] \tag{39}$$

as a power series in ε, we must successively have

$$\begin{cases} FY_0 = -G, \\ FY_1 = \dot{Y}_0 - h(Y_0,t,0), \\ \text{etc.} \end{cases} \tag{40}$$

The first equation is, of course, the reduced equation (32). Manipulating with the projections P and Q and using the invertibility of S allows us to solve for PY_0 as a linear function of QY_0, i.e.,

$$PY_0 = -E_1'S^{-1}E_1(F(QY_0) - G) \tag{41}$$

and similarly for later PY_j's as a function of QY_j. Simultaneously, $E_2F = 0$ implies that $QF = 0$, so consistency of the reduced and later equations requires that Q multiplied by the right hand side equals zero. Thus

$$(QY_0)^{\cdot} = \dot{Q}Y_0 + Qh(Y_0,t,0) \tag{42}$$

and since $Y_0 = PY_0 + QY_0$, (41) implies an initial value problem for QY_0. Using

$$Q(0)Y_0(0) = Q(0)y(0) \tag{43}$$

uniquely implies QY_0 and, thereby, Y_0. Since $P(0)Y_0(0)$ cannot be prescribed, the need for an initial boundary layer correction for $P(0)\Pi_0(0)$ is clear. Further terms follow analogously. The combined algebraic and differential equation approach allows numerical solution of these problems in cases where the usual stiff equation routines break down. Numerical work is being done with Joseph Flaherty of Rensselaer Polytechnic Institute, and will be reported soon.

Among many generalizations of the cheap control problem, consider problems for bounded scalar controls, say $|u| \leq m$. Then, one can have a saturated bang-bang control with $|u| = m$ and even an infinite number of switchings.

If we generalize (5) by considering the singular example

$$\begin{cases} \dot{x} = u, \quad x(0) = 1 \\ J(0) = \frac{1}{2}\int_0^1 x^2(t)dt, \quad |u| \leq m, \end{cases} \tag{44}$$

we obtain the optimal solution (for $m > 1$)

$$u = \begin{cases} -m, & 0 \le t < 1/m \\ 0, & t > 1/m \end{cases} \tag{45}$$

corresponding to the solution $u = -\delta$ obtained for $m = \infty$. Our singular perturbation analysis indicates (but does not prove) that for singular arcs of order one, the optimal control is initially saturated before transfer to a singular arc (cf. Flaherty and O'Malley (1977)). For Case L problems, $L > 1$, the optimal control usually switches infinitely often before reaching the singular arc. Nonetheless, for many problems, our analysis suggests how to obtain a near-optimal L-switch solution.

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These lecture notes are intended to provide an elementary account of some of the recent mathematical effort in applying singular perturbations theory to optimal control problems, to demonstrate the practical importance of this asymptotic technique to current engineering studies, and to suggest several open problems needing further research. Readers are referred to the survey article by Kokotovic, O'Malley, and Sannuti for a discussion of related topics and for additional references.

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